

CONORMAL AND PIECEWISE SMOOTH SOLUTIONS TO QUASILINEAR WAVE EQUATIONS

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ABSTRACT. In this paper, we show first that if a solution u of the equation $P_2(t, x, u, Du, D)u = f(t, x, u, Du)$, where $P_2(t, x, u, Du, D)$ is a second order strictly hyperbolic quasilinear operator, is conormal with respect to a single characteristic hypersurface Σ of P_2 in the past and Σ is smooth in the past, then Σ is smooth and u is conormal with respect to Σ for all time. Second, let Σ_0 and Σ_1 be characteristic hypersurfaces of P_2 which intersect transversally and let $\Gamma = \Sigma_0 \cap \Sigma_1$. If Σ_0 and Σ_1 are smooth in the past and u is conormal with respect to $\{\Sigma_0, \Sigma_1\}$ in the past, then Γ is smooth, and u is conormal with respect to $\{\Sigma_0, \Sigma_1\}$ locally in time outside of Γ , even though Σ_0 and Σ_1 are no longer necessarily smooth across Γ . Finally, we show that if $u(0, x)$ and $\partial_t u(0, x)$ are in an appropriate Sobolev space and are piecewise smooth outside of Γ , then u is piecewise smooth locally in time outside of $\Sigma_0 \cup \Sigma_1$.

1. INTRODUCTION

Let $u(t, x) \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$, be a solution of the quasilinear equation

$$(1.1) \quad P_2(t, x, u, Du, D)u = f(t, x, u, Du),$$

where

$$(1.2) \quad P_2(t, x, u, Du, D) \equiv \left(\partial_t^2 - \sum_{\substack{(i,j) \neq (0,0) \\ i,j=0}}^n a_{ij}(t, x, u, Du) \partial_{x_i} \partial_{x_j} \right)$$

is strictly hyperbolic with respect to $\{t = \text{constant}\}$, f is a smooth function of its arguments and (a_{ij}) is symmetric.

In this paper, we consider the regularity of solutions which are assumed to be conormal in the past with respect to a characteristic hypersurface, or a pair of characteristic hypersurfaces, or initial data conormal with respect to the intersection of a pair of characteristic hypersurfaces, and piecewise smoothness of solutions which are assumed to be piecewise smooth at $\{t = 0\}$. The conormal cases for nonlinear problems had been treated previously by Alinhac [1] and [2]. He uses the theory of paradifferential operators and an extension to

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characteristic-coordinate changes. For the second order quasilinear case, he treated H^s , $s > (n+1)/2 + 9/2$, conormal regularity with respect to a single characteristic hypersurface and H^s , $s > (n+1)/2 + 5$, conormal regularity with respect to a pair of transverse characteristic hypersurfaces. Here we use non-smooth vector fields in the given coordinates, with the regularity of the vector fields improved inductively, and the natural associated energy inequalities. Also we treat $C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > n/2 + 4$, conormal regularity with respect to both a single characteristic hypersurface and a pair of transverse characteristic hypersurfaces. The piecewise smooth case for semilinear problems had been considered by Rauch and Reed [15].

In section 1, we prove first a version of the Gagliardo-Nirenberg inequalities, which takes an important role in the proofs of various lemmas and theorems. Then we establish energy inequalities for the equations $P_2 u = f(Du)$ and $P_2 v = f_1 Dv + f_2$, where f_1 and f_2 are in the space $C(\mathbf{R}; H_{\text{loc}}^{s-5}(\mathbf{R}^n))$. Also we consider the first order nonlinear equation satisfied by the defining function $\varphi(t, x')$ of a characteristic hypersurface of P_2 . In order to treat this equation, we study the regularity of compositions of the form $v(t, \varphi(t, x'), x')$ with $v(t, x_1, x')$ and $\varphi(t, x')$ assumed to have limited Sobolev regularity.

In section 2, we prove first that if $u(t, x)$ is conormal with respect to a single characteristic hypersurface Σ (which is locally given by $\{x_1 = \varphi(t, x')\}$) of P_2 in the past and Σ is smooth in the past, then Σ is smooth and u is conormal with respect to Σ for all time. Analogous results hold locally in time, subject to the constraints of finite propagation speed. In order to prove conormal regularity, commutator arguments along with the energy inequality and the result on the regularity of compositions are then applied. Next, we consider the conormal regularity of u with respect to a pair of transverse characteristic hypersurfaces of P_2 . Let Σ_0 and Σ_1 be characteristic hypersurfaces of P_2 which intersect transversally and let $\Gamma = \Sigma_0 \cap \Sigma_1$. If Σ_0 and Σ_1 are smooth in the past and u is conormal with respect to $\{\Sigma_0, \Sigma_1\}$ in the past, then Γ is smooth, and u is conormal with respect to $\{\Sigma_0, \Sigma_1\}$ locally in time outside of Γ , even though Σ_0 and Σ_1 are no longer necessarily smooth across Γ . Examples of surfaces which are not smooth across Γ in one-space dimension are presented in Messer [9]. In proving this property, we consider appropriate regions near Γ , and apply commutator and induction arguments involving vector fields with nonsmooth coefficients as in the proof of Theorem 3.2, as well as finite propagation speed. Finally, we prove that if $u(0, x)$ and $u_t(0, x)$ are conormal with respect to Γ , then Σ_0 and Σ_1 are smooth and u is conormal with respect to $\{\Sigma_0, \Sigma_1\}$ locally in the future. The proof is analogous to the case of two characteristic hypersurfaces.

In section 3, we prove the piecewise smoothness of the solution u of (1.1) assuming piecewise smoothness initially. Then after a change of coordinates, we may write

$$\begin{aligned} P_2 &= Q_+ D_0 + \cdots \\ &= Q_- D_1 + \cdots, \end{aligned}$$

where Q_+ and Q_- are tangential vector fields to Σ_0 and Σ_1 respectively. Let \mathcal{O} be an open neighborhood of the origin and let $\mathcal{O}_0 = \mathcal{O} \cap \{t = 0\}$. Let Σ_0 , Σ_1 and Γ be as above. With a slightly changed time coordinate, we may assume, by strict hyperbolicity of P_2 and the hypothesis of Theorem 3.8

since Γ is smooth, that Γ divides \mathcal{O}_0 into two connected components and $\Sigma_0 \cup \Sigma_1$ divides \mathcal{O} in four connected components. Suppose that $u(0, x) \in H_{\text{loc}}^s(\mathbf{R}^n)$ and $u_t(0, x) \in H_{\text{loc}}^{s-1}(\mathbf{R}^n)$ and both are piecewise smooth outside of Γ . Then u is piecewise smooth locally in time. The proof involves Theorem 3.8 and repeated induction arguments together with a modification of the result of Rauch-Reed in the semilinear case: Suppose that $A(t, x)$ and $F(t, x)$ are piecewise continuous outside of $\Sigma_0 \cup \Sigma_1$. Then the initial value problem for the linear system

$$Q_+ z + Az = F$$

has a unique piecewise continuous solution on Σ_0 if $z(0, x)$ is piecewise continuous outside of Γ .

2. PRELIMINARIES

2.1. The energy inequality. In this section we will prove appropriate energy inequalities for equation (1.1). We note that strictly hyperbolicity of $P_2(t, x, u, Du, D)$ implies the following inequality:

$$(2.1) \quad |\xi|^2 \leq C(\xi_0^2 + \sum_{i,j=1}^n a_{ij}(t, x, u, Du) \xi_i \xi_j),$$

where C is a positive constant and $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbf{R}^{n+1}$. Throughout this thesis, we use

$$\sum_{(i,j) \neq (0,0)}^n \quad \text{instead of} \quad \sum_{\substack{(i,j) \neq (0,0) \\ i,j=0}}^n \quad \text{for notational convenience.}$$

Before we prove the energy inequalities we need the following lemmas. For the proof of Schauder's lemma, see Rauch [13], and for the Gagliardo-Nirenberg inequalities, see Nirenberg [11].

Lemma 2.1 (Schauder's lemma). *If $u, v \in H^s(\mathbf{R}^n)$ and $s > \frac{n}{2}$, then $uv \in H^s(\mathbf{R}^n)$ and $\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s}$.*

Lemma 2.2 (Gagliardo-Nirenberg inequalities). *Let $u \in L^q(\mathbf{R}^n)$ and its derivative of order m , $D^m u$, belong to $L^r(\mathbf{R}^n)$, $1 \leq q, r \leq \infty$. For the derivatives $D^j u$, $0 \leq j < m$, the following inequalities hold:*

$$(2.2) \quad \|D^j u\|_p \leq \text{constant} \|D^m u\|_r^a \|u\|_q^{1-a},$$

where $1/p = j/n + a(1/r - m/n) + (1-a)/q$, for all a in the interval $j/m \leq a \leq 1$ (the constants depending only on n, m, j, q, r and a) with the following exceptional cases:

- (1) If $j = 0$, $rm < n$, $q = \infty$ then we make the additional assumption that either u tends to 0 at infinity or $u \in L^{\tilde{q}}(\mathbf{R}^n)$ for some finite $\tilde{q} > 0$.
- (2) If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then (2.2) holds only for a satisfying $j/m \leq a < 1$.

Throughout this paper we treat the case in which the regularity indices s and s' are integers; analogous results hold in the general case.

Lemma 2.3. *Let $0 \leq s' \leq s$ and suppose that $w \in L^\infty(\mathbf{R}^n) \cap H^s(\mathbf{R}^n)$. Then for $|\alpha| = s'$, it follows that $D^\alpha w \in L^{2p}(\mathbf{R}^n)$ and*

$$\|D^\alpha w\|_{L^{2p}} \leq C(\|w\|_{L^\infty})^{1-\frac{1}{p}}(\|w\|_{H^s})^{\frac{1}{p}},$$

where $p = s/s'$ and C is a constant depending only on s , s' and n .

Proof. Let $D^{s'}$ stands for any derivative of order s' . For $s' = s$, we have $p = 1$ and since $w \in H^s(\mathbf{R}^n)$, $\|D^s w\|_{L^2} \leq C\|w\|_{H^s}$. For $0 \leq s' < s$, by Lemma 2.2 for $q = \infty, r = 2$ and $m = s$,

$$\|D^{s'} w\|_{L^{2p}} \leq C\|D^s w\|_{L^2}^a \|w\|_{L^\infty}^{1-a},$$

where C is a constant depending only on s , s' and n , also a is such that

$$(2.3) \quad \frac{1}{2p} = \frac{s'}{n} + a \left(\frac{1}{2} - \frac{s}{n} \right) \quad \text{with } \frac{s'}{s} \leq a < 1.$$

From (2.3), we have for $a = s'/s = 1/p$,

$$\|D^{s'} w\|_{L^{2p}} \leq C(\|w\|_{L^\infty})^{1-\frac{1}{p}}(\|w\|_{H^s})^{\frac{1}{p}}.$$

Corollary 2.4. *Suppose that $w \in L^\infty_{\text{loc}}(\mathbf{R}^n) \cap H^s_{\text{loc}}(\mathbf{R}^n)$ for $s \geq 0$. If $f \in C^\infty(\mathbf{R})$, then $f(w) \in H^s_{\text{loc}}(\mathbf{R}^n)$.*

Proof. If $|\beta| = s$, then by the chain rule and the Leibniz formula, $D^\beta(f(w))$ may be written as a sum of terms of the form $g_\beta(w)(D^{\alpha_1} w) \cdots (D^{\alpha_m} w)$ with g_β smooth and $\alpha_1 + \cdots + \alpha_m = \beta$. Since $g_\beta(w) \in L^\infty_{\text{loc}}(\mathbf{R}^n)$ and $D^{\alpha_k} w \in L^r_{\text{loc}}(\mathbf{R}^n)$, where $r = \frac{2s}{|\alpha_k|}$, by Lemma 2.3, Hölder's inequality implies that the use of the chain rule was justified, and since $(|\alpha_1| + \cdots + |\alpha_m|)/2s = 1/2$, $g_\beta(w)(D^{\alpha_1} w) \cdots (D^{\alpha_m} w) \in L^2_{\text{loc}}(\mathbf{R}^n)$.

Now we state and prove the energy inequalities for equation (1.1).

Theorem 2.5 (Energy Inequality). *Let $P_2(t, x, u, Du, D)$ be a partial differential operator of order 2 on \mathbf{R}^{n+1} , as in (1.2), strictly hyperbolic with respect to the planes $t = \text{constant}$ and let $u(t, x) \in C(\mathbf{R}; H^s_{\text{loc}}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-1}_{\text{loc}}(\mathbf{R}^n))$, $s > \frac{n}{2} + 2$, satisfy the equation*

$$(2.4) \quad P_2(t, x, u, Du, D)u(t, x) = f(t, x, u, Du),$$

where f is a smooth function of its arguments. If $u(0, x) \in H^s_{\text{loc}}(\mathbf{R}^n)$ and $u_t(0, x) \in H^{s-1}_{\text{loc}}(\mathbf{R}^n)$, then, for all t , $u(t, x) \in H^s_{\text{loc}}(\mathbf{R}^n)$. Moreover, if u has compact support in x for each time t , then we have

$$(2.5) \quad \|u(t, x)\|_{H^s(\mathbf{R}^n)} \leq C_t(\|u(0, x)\|_{H^s(\mathbf{R}^n)} + \|u_t(0, x)\|_{H^{s-1}(\mathbf{R}^n)})$$

for all t , where $C_t = C(t)$ is independent of u and f .

Proof. By finite propagation speed and an analysis local in time, it may be assumed that u has compact support in x for each t . Let $|\alpha| = s - 1$. We apply $D^\alpha \equiv \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ to equation (2.4) obtaining an equation, by the Leibniz formula,

$$(2.6) \quad P_2(D^\alpha u) = D^\alpha f + \sum_{\beta < \alpha} \sum_{(i, j) \neq (0, 0)}^n C_\beta(D^{\alpha-\beta} a_{ij})(D^\beta u_{x_i x_j}) \equiv g,$$

where C_β are constants depending only on multiindices β . The use of the Leibniz formula is justified below, using Lemma 2.3. Let $w(t, x) = D^\alpha u(t, x)$. Then (2.6) can be written as

$$(2.7) \quad P_2 w \equiv w_{tt} - \sum_{(i,j) \neq (0,0)}^n a_{ij} (Du) w_{x_i x_j} = g.$$

Let $E(t)$ be the energy for equation (2.7) defined by

$$(2.8) \quad E^2(t) = \int (w^2(t, x) + w_t^2(t, x) + \sum_{i,j=1}^n a_{ij} (Du) w_{x_i}(t, x) w_{x_j}(t, x)) dx.$$

By differentiating (2.8) with respect to t and integrating by parts, we have

$$\begin{aligned} 2E(t) \frac{dE(t)}{dt} &= \int \left(2ww_t + 2w_t w_{tt} + \sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial t} \right) w_{x_i} w_{x_j} + 2 \sum_{i,j=1}^n a_{ij} w_{x_i t} w_{x_j} \right) dx \\ &= \int \left(2ww_t + 2w_t w_{tt} - 2 \sum_{i,j=1}^n a_{ij} w_t w_{x_i x_j} - 2 \sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial x_i} \right) w_{x_j} w_t \right) dx \\ &\quad + \sum_{i,j=1}^n \int \left(\frac{\partial a_{ij}}{\partial t} \right) w_{x_i} w_{x_j} dx \\ &= 2 \int w_t \left(w + w_{tt} - \sum_{(i,j) \neq (0,0)}^n a_{ij} w_{x_i x_j} \right) dx + 4 \int \sum_{j=1}^n a_{0j} w_{tx_j} w_t dx \\ &\quad - 2 \sum_{i,j=1}^n \int \left(\frac{\partial a_{ij}}{\partial x_i} \right) w_{x_j} w_t dx + \sum_{i,j=1}^n \int \left(\frac{\partial a_{ij}}{\partial t} \right) w_{x_i} w_{x_j} dx. \end{aligned}$$

We note that

$$4 \int \sum_{j=1}^n a_{0j} w_{tx_j} w_t dx = -4 \int \sum_{j=1}^n \left(\frac{\partial a_{0j}}{\partial x_j} \right) w_t w_t dx - 4 \int \sum_{j=1}^n a_{0j} w_{tx_j} w_t dx,$$

so that

$$4 \int \sum_{j=1}^n a_{0j} w_{tx_j} w_t dx = -2 \int \sum_{j=1}^n \left(\frac{\partial a_{0j}}{\partial x_j} \right) w_t w_t dx.$$

It follows from (2.7) that

$$(2.9) \quad \begin{aligned} E(t) \frac{dE(t)}{dt} &= \int w_t (w + g) dx - \sum_{i,j=1}^n \int \left(\frac{\partial a_{ij}}{\partial x_j} \right) w_{x_i} w_t dx \\ &\quad + \frac{1}{2} \int \sum_{i,j=1}^n \left(\frac{\partial a_{ij}}{\partial t} \right) w_{x_i} w_{x_j} dx - \int \sum_{j=1}^n \left(\frac{\partial a_{0j}}{\partial x_j} \right) w_t w_t dx. \end{aligned}$$

By letting $\xi = \nabla w = (w_t, w_{x_1}, \dots, w_{x_n})$ in (2.1), we have

$$w_t^2 + w_{x_1}^2 + \dots + w_{x_n}^2 \leq C \left(w_t^2 + \sum_{i,j=1}^n a_{ij} (Du) w_{x_i} w_{x_j} \right)$$

and so, by (2.8),

$$\int (w_t^2 + \sum_{i,j=1}^n a_{ij}(Du) w_{x_i} w_{x_j}) dx \leq CE^2(t).$$

Now we estimate the third term in (2.9). Hölder's and Schwarz's inequalities imply that

$$\begin{aligned} \sum_{i,j=1}^n \int \left(\frac{\partial a_{ij}(Du)}{\partial t} \right) w_{x_i} w_{x_j} dx &\leq \sum_{i,j=1}^n \int \left| \left(\frac{\partial a_{ij}(Du)}{\partial t} \right) w_{x_i} w_{x_j} \right| dx \\ (2.10) \quad &\leq \sum_{i,j=1}^n \left(\left\| \frac{\partial a_{ij}(Du)}{\partial t} \right\|_{L^\infty} \int |w_{x_i} w_{x_j}| dx \right) \\ &\leq C(t) \sum_{i,j=1}^n \left(\int w_{x_i}^2 dx \right)^{\frac{1}{2}} \left(\int w_{x_j}^2 dx \right)^{\frac{1}{2}} \\ &\leq C(t) E^2(t), \end{aligned}$$

since $D^2 u(t, x) \in H^{s-2}(\mathbf{R}^n) \subset L^\infty(\mathbf{R}^n)$, for all t , and $s - 2 > \frac{n}{2}$. Similarly, for the second and fourth terms, we have

$$\begin{aligned} \sum_{i,j=1}^n \int \left(\frac{\partial a_{ij}(Du)}{\partial x_j} \right) w_{x_i} w_t dx &\leq \sum_{i,j=1}^n \left(\left\| \frac{\partial a_{ij}(Du)}{\partial x_j} \right\|_{L^\infty(\mathbf{R}^n)} \int |w_{x_i} w_t| dx \right) \\ (2.11) \quad &\leq C(t) \sum_{i=1}^n \left(\int w_{x_i}^2 dx \right)^{\frac{1}{2}} \left(\int w_t^2 dx \right)^{\frac{1}{2}} \\ &\leq C(t) E^2(t), \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n \int \left(\frac{\partial a_{0j}(Du)}{\partial x_j} \right) w_t w_t dx &\leq \sum_{j=1}^n \left(\left\| \frac{\partial a_{0j}(Du)}{\partial x_j} \right\|_{L^\infty(\mathbf{R}^n)} \int w_t^2 dx \right) \\ (2.12) \quad &\leq C(t) E^2(t). \end{aligned}$$

For the first term, Schwarz's inequality yields that

$$\begin{aligned} \int w_t(w + g) dx &= \int w_t w dx + \int (w_t g) dx \\ (2.13) \quad &\leq C(t) E^2(t) + \int |w_t g| dx. \end{aligned}$$

It remains to estimate $\int |w_t g| dx$.

We will show that for $1 \leq l \leq s - 1$, $(D^l a_{ij}(Du))(D^{s+1-l} u) \in L^2(\mathbf{R}^n)$. By the chain rule and the Leibniz formula, $D^l a_{ij}(Du)$ may be written as a sum of terms of the form $b_\gamma(Du)(D^{\alpha_1}(Du)) \cdots (D^{\alpha_m}(Du))$ with b_γ smooth and $|\alpha_1| + \cdots + |\alpha_m| = l$, where $|\alpha_k| \geq 1$ for $k = 1, \dots, m$. Let $D^2 u(t, x) = v(t, x)$. Then $v \in L^\infty(\mathbf{R}^n) \cap H^{s-2}(\mathbf{R}^n)$ since $s - 2 > n/2$, and $D^l a_{ij}(Du)$ is written as a sum of terms of the form $b_\gamma(Du)(D^{\beta_1} v) \cdots (D^{\beta_m} v)$ with $|\beta_1| + \cdots + |\beta_m| \leq l - 1$, where $|\beta_k| \geq 0$ for $k = 1, \dots, m$. As we see in the

proof of Corollary 2.4, $b_\gamma(Du) \in L^\infty(\mathbf{R}^n)$ and $D^{\beta_k}v \in L^p(\mathbf{R}^n)$, where $p = \frac{2(s-2)}{|\beta_k|}$, by Lemma 2.3. Thus Hölder's inequality implies that the use of the chain rule was justified, and since $(|\beta_1| + \dots + |\beta_m|)/2(s-2) \leq (l-1)/2(s-2)$, $b_\gamma(Du)(D^{\beta_1}v) \dots (D^{\beta_m}v) \in L^q(\mathbf{R}^n)$, where $q = \frac{2(s-2)}{l-1}$. Since $D^{s+1-l}u = D^{s-1-l}v$, $D^{s+1-l}u \in L^r(\mathbf{R}^n)$, where $r = \frac{2(s-2)}{s-1-l}$, by Lemma 2.3. Therefore, by Hölder's inequality and Lemma 2.3, $(D^l a_{ij}(Du))(D^{s+1-l}u) \in L^2(\mathbf{R}^n)$ and $\|(D^l a_{ij}(Du))(D^{s+1-l}u)\|_{L^2} \leq \|D^l a_{ij}(Du)\|_{L^p} \|D^{s+1-l}u\|_{L^q} \leq C(t)\|v\|_{L^\infty}\|v\|_{H^{s-2}}$ since $1/p + 1/q = 1/2$, where $p = 2(s-2)/(l-1)$ and $q = 2(s-2)/(s-1-l)$. As we have seen in the proof of Corollary 2.4, $D^\alpha(f(Du))$ may be written as a sum of terms of the form $f_\alpha(Du)(D^{\alpha_1}(Du)) \dots (D^{\alpha_m}(Du))$ with f_α smooth, $\alpha_1 + \dots + \alpha_m = \alpha$ and $f_\alpha(Du)(D^{\alpha_1}(Du)) \dots (D^{\alpha_m}(Du)) \in L^2(\mathbf{R}^n)$. Moreover, by Lemma 2.3, $\|D^\alpha(f(Du))\|_{L^2} \leq C(t)\|Du\|_{H^{s-1}} \leq C(t)\|u\|_{H^s}$.

By Schwarz's inequality, Minkowski's inequality and the facts above,

$$\begin{aligned} \int (w_t g) dx &= \int w_t \left(D^\alpha f + \sum_{\beta < \alpha} \sum_{(i,j) \neq (0,0)}^n C_\beta (D^{\alpha-\beta} a_{ij})(D^\beta u_{x_i x_j}) \right) dx \\ &\leq \left(\int w_t^2 dx \right)^{\frac{1}{2}} \left(\int (D^\alpha f)^2 dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int w_t^2 dx \right)^{\frac{1}{2}} \left[\sum_{\beta < \alpha} \sum_{(i,j) \neq (0,0)}^n C_\beta \left(\int (D^{\alpha-\beta} a_{ij})^2 (D^\beta u_{x_i x_j})^2 dx \right)^{\frac{1}{2}} \right] \\ &\leq C(t)E(t) (\|D^\alpha f\|_{L^2} + \|v\|_{L^\infty}\|v\|_{H^{s-2}}) \\ &\leq C(t)E(t)(\|u\|_{H^s}) \\ &\leq C(t)E^2(t), \end{aligned}$$

and so

$$(2.14) \quad \int w_t(w + g) dx \leq C(t)E^2(t).$$

From (2.10), (2.11), (2.12), (2.14) and by dividing (2.9) by $E(t)$, we have

$$(2.15) \quad \frac{dE(t)}{dt} \leq C(t)E(t).$$

By applying Gronwall's inequality to (2.15),

$$(2.16) \quad E(t) \leq \tilde{C}(t)E(0), \quad \text{where } d\tilde{C}(t) = \exp \left(\int_0^t C(\lambda) d\lambda \right).$$

Thus, for given t ,

$$\|u(t, x)\|_{H^s} \leq C(t) (\|u(0, x)\|_{H^s} + \|u_t(0, x)\|_{H^{s-1}}).$$

From the theorem above, we have the following corollary which will be used in the proofs of Theorem 3.2 and Theorem 3.8.

Corollary 2.6. *Let $P_2(t, x, u, Du, D)$ be the same as above and assume that $u(t, x) \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$. Let $v(t, x)$ satisfy the equation*

$$(2.17) \quad P_2(t, x, u, Du, D)v(t, x) = f_1(t, x)Dv(t, x) + f_2(t, x),$$

where $f_1(t, x), f_2(t, x) \in C(\mathbf{R}; H_{\text{loc}}^{s-5}(\mathbf{R}^n))$. If $v(0, x) \in H_{\text{loc}}^{s-4}(\mathbf{R}^n)$ and $v_t(0, x) \in H_{\text{loc}}^{s-5}(\mathbf{R}^n)$, then $v(t, x) \in H_{\text{loc}}^{s-4}(\mathbf{R}^n)$ for all t . Moreover, if v has compact support in x for each time t , then we have

$$(2.18) \quad \|v(t, x)\|_{H^{s-4}} \leq C_1(t) (\|v(0, x)\|_{H^{s-4}} + \|v_t(0, x)\|_{H^{s-5}}) + C_2(t),$$

for all t , where $C_1(t), C_2(t)$ are independent of u and v .

Proof. By finite propagation speed and analysis local in time, it may assumed that v has compact support in x for each t . As we see in the proof of the energy inequality 2.5, we have

$$(2.19) \quad \begin{aligned} P_2(D^\alpha v) &= D^\alpha(f_1(t, x)(Dv)) + D^\alpha(f_2(t, x)) \\ &\quad + \sum_{\beta < \alpha} \sum_{(i,j) \neq (0,0)}^n C_\beta(D^{\alpha-\beta} a_{ij}(Du))(D^\beta v_{x_i x_j}) \\ &\equiv \tilde{g}, \end{aligned}$$

where $|\alpha| = s - 5$. Let $w(t, x) = D^\alpha v(t, x)$. Then (2.19) becomes

$$(2.20) \quad P_2 w \equiv w_t - \sum_{(i,j) \neq (0,0)}^n a_{ij}(Du) w_{x_i x_j} = \tilde{g}.$$

Let $\tilde{E}(t)$ be the energy for equation (2.20) defined as in (2.8). We will establish an *a priori* estimate on $\tilde{E}(t)$. As we see in the proof of energy inequality 2.5, all estimates except $\int |w_t \tilde{g}| dx$ are valid, since $s > \frac{n}{2} + 4$. So it remains to estimate $\int |w_t \tilde{g}| dx$. First we show that, for $1 \leq l \leq s-5$, $(D^l a_{ij}(Du))(D^{s+1-l} v) \in L^2(\mathbf{R}^n)$. From the proof of energy inequality 2.5, $D^l a_{ij}(Du) \in L^{2p}(\mathbf{R}^n)$, where $p = (s-2)/(l-1)$ and $\|D^l a_{ij}(Du)\|_{L^{2p}} \leq C(t) \|u\|_{H^s} \leq C(t)$. Since $v \in L^\infty(\mathbf{R}^n) \cap H^{s-4}(\mathbf{R}^n)$, $s > \frac{n}{2} + 4$, $D^{s-3-l} v \in L^{2q}(\mathbf{R}^n)$, where $q = (s-4)/(s-3-l)$ and $\|D^{s-3-l} v\|_{L^{2q}} \leq C(t) \|v\|_{H^{s-4}}^{1-\frac{1}{q}} \leq C(t) \tilde{E}(t)$, by Lemma 2.3. Therefore, $(D^l a_{ij}(Du))(D^{s+1-l} v) \in L^2(\mathbf{R}^n)$, since

$$(l-1)/2(s-2) + (s-3-l)/2(s-4) \leq 1/2,$$

and

$$\begin{aligned} \|(D^l a_{ij}(Du))(D^{s-3-l} v)\|_{L^2} &\leq \|D^l a_{ij}(Du)\|_{L^{2p}} \|D^{s-3-l} v\|_{L^{2q}} \\ &\leq C(t) \tilde{E}(t). \end{aligned}$$

Next we estimate $D^\alpha(f_1(t, x)(Dv))$. By the Leibniz formula, $D^\alpha(f_1(t, x)(Dv))$ may be written as a sum of terms of the form $(D^{\alpha_1} f_1)(D^{\alpha_2}(Dv))$ with $\alpha_1 + \alpha_2 = \alpha$. By Schwarz's inequality and Minkowski's inequality, $D^\alpha(f_1(t, x)(Dv)) \in L^2(\mathbf{R}^n)$ and $\|D^\alpha(f_1(t, x)(Dv))\|_{L^2} \leq \|f_1\|_{H^{s-5}} \tilde{E}(t)$.

Therefore, by the facts above,

$$\begin{aligned} \int (w_t \tilde{g}) dx &= \int w_t [D^\alpha(f_1(t, x)(Dv)) + D^\alpha(f_2(t, x))] \\ &\quad + \int w_t \left[\sum_{\beta < \alpha} \sum_{(i,j) \neq (0,0)}^n C_\beta(D^{\alpha-\beta} a_{ij}(Du))(D^\beta v_{x_i x_j}) \right] dx \\ &\leq C(t) \tilde{E}(t) (\|f_1\|_{H^{s-5}} \tilde{E}(t) + \|f_2\|_{H^{s-5}} + \tilde{E}(t)), \end{aligned}$$

and so

$$(2.21) \quad \frac{d\tilde{E}(t)}{dt} \leq C(t) (\|f_1\|_{H^{s-5}}\tilde{E}(t) + \|f_2\|_{H^{s-5}}).$$

Therefore, by applying Gronwall's inequality to (2.21),

$$\tilde{E}(t) \leq e^{h(t)} \left(\tilde{E}(0) + \int_0^t C(\tau) \|f_2(\tau)\|_{H^{s-5}} e^{-h(\tau)} d\tau \right),$$

where

$$h(t) = \int_0^t C(\tau) \|f_1(\tau)\|_{H^{s-5}} d\tau \quad \text{and} \quad f_i(\tau) \equiv f_i(\tau, x), \quad i = 1, 2.$$

Let $C_1(t) = e^{h(t)}$ and $C_2(t) = e^{h(t)} \int_0^t C(\tau) \|f_2(\tau)\|_{H^{s-5}} e^{-h(\tau)} d\tau$. Then we have

$$\tilde{E}(t) \leq C_1(t)\tilde{E}(0) + C_2(t),$$

which implies that $\tilde{E}(t)$ is finite for all time t , since by assumption it is finite at $t = 0$. Therefore,

$$\|v(t, x)\|_{H^{s-4}} \leq C_1(t) (\|v(0, x)\|_{H^{s-4}} + \|v_t(0, x)\|_{H^{s-5}}) + C_2(t).$$

2.2. First order nonlinear equations. Suppose that a characteristic hypersurface for P_2 is given by $\{x_1 = \varphi(t, x')\}$, where $x' = (x_2, \dots, x_n) \in \mathbf{R}^{n-1}$. Then φ satisfies the first order nonlinear equation

$$(2.22) \quad (\varphi_t)^2 + 2(a_{01} - \sum_{j=2}^n a_{0j}\varphi_{x_j})\varphi_t = a_{11} - 2\sum_{j=2}^n a_{1j}\varphi_{x_j} + \sum_{i,j=2}^n a_{ij}\varphi_{x_i}\varphi_{x_j}.$$

The coefficients in (2.22) are evaluated at $x_1 = \varphi(t, x')$, and thus the regularity of functions of the form $a(Du(t, \varphi(t, x'), x'))$ needs to be examined.

We begin by studying the regularity of compositions of the form $v(\varphi(x'), x')$ with v and φ assumed to have limited Sobolev regularity. Even if $\varphi(x')$ is smooth, functions of the form $v(\varphi(x'), x')$ will in general have Sobolev regularity of order $1/2$ lower than that of $v(x_1, x')$. When $\varphi(x')$ is nonsmooth, the regularity of $v(\varphi(x'), x')$ will not in general be greater than that of $\varphi(x')$. In order to obtain norm estimates on the regularity of $v(\varphi(x'), x')$ that are linear in the norm of $\varphi(x')$, we will assume that the Sobolev regularity of $v(x_1, x')$ is at least one order greater than that of $\varphi(x')$.

Lemma 2.7. *Let $v(x_1, x') \in H_{\text{loc}}^{s+1}(\mathbf{R}^n)$ for $s > \frac{n}{2} + 1$. Suppose that $\varphi(x') \in H_{\text{loc}}^{s'}(\mathbf{R}^{n-1})$, $1 \leq s' \leq s$, and that $D\varphi(x') \in L^\infty(\mathbf{R}^{n-1})$. Then*

$$v(\varphi(x'), x') \in H_{\text{loc}}^{s'}(\mathbf{R}^{n-1}).$$

If v and φ have compact support, then $\|v(\varphi(x'), x')\|_{H^{s'}} \leq C\|\varphi\|_{H^{s'}}$ with C depending only on s, s', n , the size of the supports, $\|v(x_1, x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.

Proof. We may assume without loss of generality that v and φ have compact support. Let w represent the vector consisting of all derivatives of $\partial_{x_1}v$ (i.e., $w = (\partial_{x_1}(\partial_{x_1}v), \dots, \partial_{x_n}(\partial_{x_1}v))$). Then $w \in H^{s-1}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ since $s-1 > \frac{n}{2}$, and therefore $D^\alpha w \in L^r(\mathbf{R}^n)$, where $r = \frac{2(s-1)}{|\alpha|}$, for $0 \leq |\alpha| \leq s-1$ by

Lemma 2.3. Thus $D^\beta \partial_{x_1} v \in L^{r'}(\mathbf{R}^n)$, where $r' = \frac{2(s-1)}{|\beta|-1}$, for $1 \leq |\beta| \leq s$, and since

$$(2.23) \quad (D^\beta v)(\varphi(x'), x') = \int_{-\infty}^{\varphi(x')} D^\beta \partial_{x_1} v(x_1, x') dx_1,$$

it follows from Minkowski's inequality that $(D^\beta v)(\varphi(x'), x') \in L^r(\mathbf{R}^{n-1})$, where $r = \frac{2(s-1)}{|\beta|-1}$, with norm depending only on the size of the supports, $\|v(x_1, x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.

Next, notice that $D\varphi \in H^{s'-1}(\mathbf{R}^{n-1}) \cap L^\infty(\mathbf{R}^{n-1})$, and therefore $D^\beta \varphi \in L^{r'}(\mathbf{R}^{n-1})$, where $r' = \frac{2(s'-1)}{|\beta|-1}$, for $1 \leq |\beta| \leq s'$, by Lemma 2.3. The chain rule and the Leibniz formula imply that, for $1 \leq |\gamma| \leq s'$, $D^\gamma(v(\varphi(x'), x'))$ may be written as a sum of terms of the form

$$(D^{\beta_0} D^m v)(\varphi(x'), x') (D^{\beta_1} \varphi(x')) \cdots (D^{\beta_m} \varphi(x')),$$

where D^m stands for a derivative of order m with respect to x_1 with $0 \leq m \leq s'$ and $\beta_0 + \beta_1 + \cdots + \beta_m = \gamma$, $1 \leq |\beta_k| \leq s'$ for $k = 0, 1, \dots, m$. If $m = 0$, then $\gamma = \beta_0$, and so $(D^{\beta_0} v)(\varphi(x'), x') \in L^q(\mathbf{R}^{n-1}) \subset L^2(\mathbf{R}^{n-1})$, where $q = \frac{2(s-1)}{|\beta_0|-1}$, with $\|(D^{\beta_0} v)(\varphi(x'), x')\|_{L^2} \leq C$, where C depends only on the size of supports, $s, s', n, \|v(x_1, x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$. Therefore, we may assume that $m \geq 1$. The preceding estimates and Hölder's inequality imply that the use of the chain rule was justified, and

$$(D^{\beta_0} D^m v)(\varphi(x'), x') (D^{\beta_1} \varphi(x')) \cdots (D^{\beta_m} \varphi(x')) \in L^2(\mathbf{R}^{n-1})$$

since

$$\frac{|\beta_0| + (m-1)}{2(s-1)} + \frac{(|\beta_1| - 1) + \cdots + (|\beta_m| - 1)}{2(s'-1)} \leq \frac{|\gamma| - 1}{2(s'-1)} \leq \frac{1}{2}.$$

Therefore $D^\gamma(v(\varphi(x'), x')) \in L^2(\mathbf{R}^{n-1})$ for $1 \leq |\gamma| \leq s'$. Moreover, by Lemma 2.3, $\|D^\gamma(v(\varphi(x'), x'))\|_{L^2}$ is bounded up to an appropriate constant by

$$(\|D\varphi\|_{H^{s'-1}})^{\frac{|\beta_1|-1}{s'-1}} \cdots (\|D\varphi\|_{H^{s'-1}})^{\frac{|\beta_m|-1}{s'-1}}.$$

Since $(|\beta_1| - 1) + \cdots + (|\beta_m| - 1) \leq |\gamma| - m \leq s' - m \leq s' - 1$, the required estimate holds.

For solutions of hyperbolic equations, it is natural to consider functions which are continuous in time, with values in appropriate Sobolev spaces in the remaining variables.

Corollary 2.8. Let $v(t, x_1, x') \in C(\mathbf{R}; H_{\text{loc}}^{s+1}(\mathbf{R}^n))$ for $s > \frac{n}{2} + 1$. Suppose $\varphi(t, x) \in C(\mathbf{R}; H_{\text{loc}}^{s'}(\mathbf{R}^{n-1}))$, $1 \leq s' \leq s$, and that $D\varphi(t, x') \in C(\mathbf{R}; L_{\text{loc}}^\infty(\mathbf{R}^{n-1}))$. Then $v(t, \varphi(t, x'), x') \in C(\mathbf{R}; H_{\text{loc}}^{s'}(\mathbf{R}^{n-1}))$.

Proof. The proof of Lemma 2.7 easily yields that

$$D^\beta \partial_{x_1} v(t, x_1, x') \in C(\mathbf{R}; L^{\frac{2(s-1)}{|\beta|-1}}(\mathbf{R}^n)) \quad \text{for } 1 \leq |\beta| \leq s.$$

Let $p = 2(s - 1)/(|\beta| - 1)$. Then, from (2.23) with t as a parameter,

$$\begin{aligned} & \left(D^\beta v \right) (t, \varphi(t, x'), x') - \left(D^\beta v \right) (t, \varphi(t_0, x'), x') \\ &= \int_{-\infty}^{\varphi(t, x')} D^\beta \partial_{x_1} v(t, x_1, x') dx_1 - \int_{-\infty}^{\varphi(t_0, x')} D^\beta \partial_{x_1} v(t, x_1, x') dx_1 \\ &= \int_{\varphi(t_0, x')}^{\varphi(t, x')} D^\beta \partial_{x_1} v(t, x_1, x') dx_1. \end{aligned}$$

It follows that

$$\begin{aligned} & \left\| \left(D^\beta v \right) (t, \varphi(t, x'), x') - \left(D^\beta v \right) (t, \varphi(t_0, x'), x') \right\|_{L^p(\mathbf{R}^{n-1})} \\ & \leq C \|\varphi(t, x') - \varphi(t_0, x')\|_{L^\infty(\mathbf{R})} \rightarrow 0 \quad \text{as } t \rightarrow t_0, \end{aligned}$$

and similarly

$$\begin{aligned} & \left\| \left(D^\beta v \right) (t, \varphi(t_0, x'), x') - \left(D^\beta v \right) (t_0, \varphi(t_0, x'), x') \right\|_{L^p(\mathbf{R}^{n-1})} \\ & \leq C \|v(t, x_1, x') - v(t_0, x_1, x')\|_{H^{s+1}(\mathbf{R}^n)} \rightarrow 0 \quad \text{as } t \rightarrow t_0. \end{aligned}$$

Therefore

$$\left(D^\beta v \right) (t, \varphi(t, x'), x') \in C(\mathbf{R}; L^p(\mathbf{R}^{n-1})) \quad \text{for } 1 \leq |\beta| \leq s.$$

We note that $D\varphi \in C(\mathbf{R}; H^{s'-1}(\mathbf{R}^{n-1})) \cap C(\mathbf{R}; L^\infty(\mathbf{R}^{n-1}))$ and therefore

$$D^\beta \varphi \in C(\mathbf{R}; L^{\frac{2(s'-1)}{|\beta|-1}}(\mathbf{R}^{n-1})) \quad \text{for } 1 \leq |\beta| \leq s',$$

by Lemma 2.3. The rest of the proof of Lemma 2.7 then easily yields continuity of the norm estimates in the parameter t .

In the proof of conormal regularity of a solution of the equation $P_2 u = f(Du)$, where f is smooth of its arguments, we will encounter situations in which regularity of φ with respect to v is greater than the case considered above.

Lemma 2.9. *Let $v(x_1, x') \in H_{\text{loc}}^{s+1}(\mathbf{R}^n)$ for $s > \frac{n}{2}$. Suppose that $\varphi(x') \in H_{\text{loc}}^{s+1}(\mathbf{R}^{n-1})$ and that $D\varphi(x') \in L^\infty(\mathbf{R}^{n-1})$. Then $v(\varphi(x'), x') \in H_{\text{loc}}^s(\mathbf{R}^{n-1})$. If v and φ have compact support, then $\|v(\varphi(x'), x')\|_{H^s} \leq C \|\varphi\|_{H^s}$ with C depending only on s, n , the size of the supports, $\|v(x_1, x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.*

Proof. We may assume without loss of generality that v and φ have compact support. Since $s > \frac{n}{2}$, $\partial_{x_1} v \in H^s(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Therefore, by Lemma 2.3, $D^\beta \partial_{x_1} v \in L^{\frac{2s}{|\beta|}}(\mathbf{R}^n)$ for $1 \leq |\beta| \leq s$. It follows from Minkowski's inequality and (2.23) that $(D^\beta v)(\varphi(x'), x') \in L^{\frac{2s}{|\beta|}}(\mathbf{R}^{n-1})$, with norm depending only on s, n , the size of the supports, $\|v(x_1, x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$.

We notice that $D\varphi \in H^s(\mathbf{R}^{n-1}) \cap L^\infty(\mathbf{R}^{n-1})$; therefore $D^\beta \varphi \in L^{\frac{2s}{|\beta|-1}}(\mathbf{R}^{n-1})$ for $1 \leq |\beta| \leq s$, by Lemma 2.3. The chain rule and the Leibniz formula imply that, for $1 \leq |\gamma| \leq s$, $D^\gamma(v(\varphi(x'), x'))$ may be written as a sum of terms of the form

$$\left(D^{\beta_0} D^m v \right) (\varphi(x'), x') \left(D^{\beta_1} \varphi(x') \right) \cdots \left(D^{\beta_m} \varphi(x') \right),$$

where D^m stands for a derivative of order m with respect to x_1 with $0 \leq m \leq s$ and $\beta_0 + \beta_1 + \dots + \beta_m = \gamma$, $1 \leq |\beta_k| \leq s$ for $k = 0, 1, \dots, m$. If $m = 0$, then $\gamma = \beta_0$, and so $(D^{\beta_0} v)(\varphi(x'), x') \in L^{\frac{2}{|\beta_0|}}(\mathbf{R}^{n-1}) \subset L^2(\mathbf{R}^{n-1})$ with $\|(D^{\beta_0} v)(\varphi(x'), x')\|_{L^2} \leq C$, where C depends only on the size of supports, s , n , $\|v(x_1, x')\|_{H^{s+1}}$ and $\|\varphi\|_{L^\infty}$. Therefore, we may assume that $m \geq 1$. The preceding estimates and Hölder's inequality imply that the use of the chain rule was justified, and

$$(D^{\beta_0} D^m v)(\varphi(x'), x') (D^{\beta_1} \varphi(x')) \dots (D^{\beta_m} \varphi(x')) \in L^2(\mathbf{R}^{n-1})$$

since

$$\frac{|\beta_0| + m}{2s} + \frac{(|\beta_1| - 1) + \dots + (|\beta_m| - 1)}{2s} \leq \frac{|\gamma|}{2s} \leq \frac{1}{2}.$$

Therefore $D^\gamma(v(\varphi(x'), x')) \in L^2(\mathbf{R}^{n-1})$ for $1 \leq |\gamma| \leq s$. Moreover, by Lemma 2.3, $\|D^\gamma(v(\varphi(x'), x'))\|_{L^2}$ is bounded up to an appropriate constant by

$$(\|D\varphi\|_{H^s})^{\frac{|\beta_1|-1}{s}} \dots (\|D\varphi\|_{H^s})^{\frac{|\beta_m|-1}{s}}.$$

Since $(|\beta_1| - 1) + \dots + (|\beta_m| - 1) \leq s - 1$, the required estimate holds.

According to Lemma 2.9, we have the following corollary similar to Corollary 2.8.

Corollary 2.10. *Let $v(t, x_1, x') \in C(\mathbf{R}; H_{\text{loc}}^{s+1}(\mathbf{R}^n))$ for $s > \frac{n}{2}$. Suppose that $\varphi(t, x) \in C(\mathbf{R}; H_{\text{loc}}^{s+1}(\mathbf{R}^{n-1}))$, and that $D\varphi(t, x') \in C(\mathbf{R}; L_{\text{loc}}^\infty(\mathbf{R}^{n-1}))$. Then $v(t, \varphi(t, x'), x') \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^{n-1}))$.*

The defining function for the characteristic hypersurface $x_1 = \varphi(t, x')$ associated with the quasilinear equation (1.1) satisfies (2.22). Thus φ_t may be expressed locally as a smooth function of $t, x', u(t, \varphi(t, x'), x')$, $Du(t, \varphi(t, x'), x')$ and $\tilde{D}\varphi(t, x')$, where $\tilde{D}\varphi$ is the x' gradient of φ . Such a function will be denoted by $f(v(t, \varphi(t, x'), x'), \tilde{D}\varphi(t, x'))$, with v representing the vector (t, x', u, Du) . From now on, we use D as total derivative and \tilde{D} as x' derivative.

Theorem 2.11. *Let $v(t, x_1, x') \in C(\mathbf{R}; H_{\text{loc}}^{s+1}(\mathbf{R}^n))$ for $s > \frac{n}{2} + 1$, and assume that $D^2\varphi(t, x') \in L_{\text{loc}}^\infty(\mathbf{R} \times \mathbf{R}^{n-1})$. Let f be a smooth function of its arguments, and suppose that $\varphi_t(t, x') = f(v(t, \varphi(t, x'), x'), \tilde{D}\varphi(t, x'))$. If $\varphi(0, x') \in H_{\text{loc}}^s(\mathbf{R}^{n-1})$, then $\varphi(t, x') \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^{n-1}))$.*

Proof. It can be assumed that the functions in question all have compact support in x' . Let $\varphi^{(s)}$ denote the vector of all x' derivatives of φ up to order s . Under the assumption that φ is smooth, we will establish an *a priori* estimate on the energy $E(t) = (\int |\varphi^{(s)}(t, x')|^2 dx')^{\frac{1}{2}}$. Standard arguments then allow the smoothness assumption to be dropped.

The chain rule and the Leibniz formula imply that there are smooth functions f, F and f_α for $|\alpha_1| + \dots + |\alpha_k| + |\alpha_{k+1}| + \dots + |\alpha_m| \leq s$, $k \geq 1$, which are then evaluated at $(v(t, \varphi(t, x'), x'), \tilde{D}\varphi(t, x'))$, such that

$$\begin{aligned} \partial_t \varphi^{(s)} &= F \tilde{D} \varphi^{(s)}(t, x') + f + \sum_{\alpha} f_{\alpha} \tilde{D}^{\alpha_1}(v(t, \varphi, x')) \\ &\quad \dots \tilde{D}^{\alpha_k}(v(t, \varphi, x')) \tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi) \dots \tilde{D}^{\alpha_m}(\tilde{D}\varphi). \end{aligned}$$

The energy satisfies $E(t)\partial_t E(t) = \int \varphi^{(s)}(t, x')\partial_t \varphi^{(s)}(t, x')dx'$, and by integration by parts,

$$\begin{aligned} \int \varphi^{(s)} F(v, \tilde{D}\varphi) (\tilde{D}\varphi^{(s)}) dx' &= \frac{1}{2} \int F(v, \tilde{D}\varphi) \tilde{D}((\varphi^{(s)})^2) dx' \\ &= -\frac{1}{2} \int \tilde{D}F(v, \tilde{D}\varphi) (\varphi^{(s)})^2 dx'. \end{aligned}$$

We note, by the chain rule, that $\tilde{D}F(v, \tilde{D}\varphi) = \sum_{\sigma} F_{\sigma}(\tilde{D}v) + G_{\sigma}(\tilde{D}^2\varphi)$, where F_{σ} and G_{σ} are smooth functions of their arguments $v, \tilde{D}\varphi$. Since $v(t, x_1, x') \in C(\mathbf{R}; H^{s+1}(\mathbf{R}^n))$ and $s > \frac{n}{2} + 1$, the Sobolev imbedding theorem and Corollary 2.8 imply that $v(t, \varphi, x') \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^n)$, $D(v(t, \varphi, x')) \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^n)$, and since $D^2\varphi(t, x') \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$, $\tilde{D}(F(v(t, \varphi, x'), \tilde{D}\varphi(t, x')))) \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$, $f(v(t, \varphi, x'), \tilde{D}\varphi(t, x')) \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$ and $f_{\alpha}(v(t, \varphi, x'), \tilde{D}\varphi(t, x')) \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$.

By Lemma 2.7 and Corollary 2.8, $v(t, \varphi(t, x'), x') \in C(\mathbf{R}; H^s(\mathbf{R}^{n-1}))$ with $\|v(t, \varphi(t, x'), x')\|_{H^s} \leq C(t)E(t)$. Therefore

$$\tilde{D}(v(t, \varphi(t, x'), x')) \in C(\mathbf{R}; H^{s-1}(\mathbf{R}^{n-1})) \cap L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$$

with similar bounds on the $H^{s-1}(\mathbf{R}^{n-1})$ norm. Hence from Lemma 2.3,

$$\tilde{D}^{\alpha_j}(v(t, \varphi(t, x'), x')) \in L^{2p_j}(\mathbf{R}^{n-1}), \quad \text{where } p_j = \frac{s-1}{|\alpha_j|-1}$$

and

$$(2.24) \quad \|\tilde{D}^{\alpha_j}(v(t, \varphi(t, x'), x'))\|_{L^{2p_j}} \leq C\|\tilde{D}v\|_{L^{\infty}}^{1-\frac{1}{p_j}} \|\tilde{D}v\|_{H^{s-1}}^{\frac{1}{p_j}} \leq C(t)(E(t))^{\frac{1}{p_j}},$$

for $1 \leq |\alpha_j| \leq s$, $j = 1, \dots, k$. Similarly,

$$(D\varphi) \in C(\mathbf{R}; H^{s-1}(\mathbf{R}^{n-1})) \cap L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1}),$$

with $\|D\varphi\|_{H^{s-1}} \leq E(t)$ and $\|D\varphi\|_{L^{\infty}} \leq C(t)$. Thus, again from Lemma 2.3,

$$\tilde{D}^{\alpha_j}(\tilde{D}\varphi(t, x')) \in L^{2q_j}(\mathbf{R}^{n-1}), \quad \text{where } q_j = \frac{s-1}{|\alpha_j|},$$

and

$$(2.25) \quad \|\tilde{D}^{\alpha_j}(\tilde{D}\varphi)\|_{L^{2q_j}} \leq C\|D\varphi\|_{L^{\infty}}^{1-\frac{1}{q_j}} \|D\varphi\|_{H^{s-1}}^{\frac{1}{q_j}} \leq C(t)(E(t))^{\frac{1}{q_j}},$$

for $0 \leq |\alpha_j| \leq s-1$, $j = k+1, \dots, m$. Therefore, by Hölder's inequality,

$$\begin{aligned} \tilde{D}^{\alpha_1}(v(t, \varphi(t, x'), x')) \cdots \tilde{D}^{\alpha_k}(v(t, \varphi(t, x'), x')) \tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi) \cdots \tilde{D}^{\alpha_m}(\tilde{D}\varphi) \\ \in C(\mathbf{R}; L^2(\mathbf{R}^{n-1})), \end{aligned}$$

since $((|\alpha_1|-1) + \dots + (|\alpha_k|-1) + |\alpha_{k+1}| + \dots + |\alpha_m|)/2(s-1) \leq (s-k)/2(s-1) \leq 1/2$. From (2.24) and (2.25),

$$\begin{aligned} (2.26) \quad &\|(\tilde{D}^{\alpha_1}v) \cdots (\tilde{D}^{\alpha_k}v) \tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi) \cdots \tilde{D}^{\alpha_m}(\tilde{D}\varphi)\|_{L^2} \\ &\leq C(t)(E(t))^{\frac{s-k}{s-1}} \leq C(t)E(t). \end{aligned}$$

By Minkowski's inequality, Schwarz's inequality and (2.26),

$$\begin{aligned}
 |E(t)\partial_t E(t)| &\leq C\|\tilde{D}F\|_{L^\infty} \int \left(\varphi^{(s)}\right)^2 dx' \\
 &\quad + C \left(\int \left(\varphi^{(s)}\right)^2 dx' \right)^{\frac{1}{2}} \left[\|f\|_{L^\infty} + \sum_{\alpha} \|f_{\alpha}(\tilde{D}^{\alpha_1} v(t, \varphi, x')) \right. \\
 &\quad \left. \cdots (\tilde{D}^{\alpha_k} v(t, \varphi, x')) \tilde{D}^{\alpha_{k+1}}(\tilde{D}\varphi) \cdots \tilde{D}^{\alpha_m}(\tilde{D}\varphi) \|_{L^2} \right] \\
 &\leq CE(t) (\|\tilde{D}F\|_{L^\infty} E(t) + \|f\|_{L^\infty} + \|f_{\alpha}\|_{L^\infty} C(t) E(t)) ,
 \end{aligned}$$

and so $|\partial_t E(t)| \leq C(t)E(t) + C(t)$. Therefore, by Gronwall's inequality, $E(t)$ is finite for all time since by assumption it is finite at $t = 0$.

We will also encounter first order equations like those in Theorem 2.11 which are linear, but with coefficients of finite regularity.

Corollary 2.12. *Let $\psi(t, x')$ satisfy the equation*

$$(2.27) \quad \psi_t = F_1(t, x')\tilde{D}\psi + F_2(t, x')\psi + F_3(t, x'),$$

where

$$F_1(t, x') \in C(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^{n-1}))$$

and

$$F_2(t, x'), F_3(t, x') \in C(\mathbf{R}; H_{\text{loc}}^{s-2}(\mathbf{R}^{n-1})), \quad s > \frac{n}{2} + 2.$$

If $\psi(0, x') \in H_{\text{loc}}^{s-2}(\mathbf{R}^{n-1})$, then $\psi(t, x') \in C(\mathbf{R}; H_{\text{loc}}^{s-2}(\mathbf{R}^{n-1}))$.

Proof. It can be assumed that the functions in question all have support in x' . Let $\psi^{(s-2)}$ denote the vector of all x' derivatives of ψ up to order $s-2$. As we see in the proof of Theorem 2.11, we will establish *a priori* estimates on the energy $E(t) = (\int |\psi^{(s-2)}(t, x')|^2 dx')^{\frac{1}{2}}$.

The chain rule and the Leibniz formula imply that for $|\alpha_1| + |\alpha_2| \leq s-1$, $1 \leq |\alpha_2| \leq s-2$ and $|\beta_1| + |\beta_2| \leq s-2$,

$$\begin{aligned}
 \partial_t \psi^{(s-2)} &= F_1(t, x')\tilde{D}\psi^{(s-2)}(t, x') + \sum_{\alpha} (\tilde{D}^{\alpha_1} F_1(t, x')) (\tilde{D}^{\alpha_2} \psi(t, x')) \\
 &\quad + \sum_{\beta} (\tilde{D}^{\beta_1} F_2(t, x')) (\tilde{D}^{\beta_2} \psi(t, x')) + \tilde{D}^{s-2} F_3(t, x').
 \end{aligned}$$

The energy satisfies $E(t)\partial_t E(t) = \int \psi^{(s-2)}(t, x')\partial_t \psi^{(s-2)}(t, x')dx'$, and by integration by parts,

$$\int \psi^{(s-2)} F_1(t, x') (\tilde{D}\psi^{(s-2)}) dx' = -\frac{1}{2} \int \tilde{D}F_1(t, x') (\psi^{(s-2)})^2 dx'.$$

Since $s > \frac{n}{2} + 2$, the Sobolev imbedding theorem implies that $\tilde{D}F_1(t, x') \in L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$.

Since $\tilde{D}F_1(t, x') \in C(\mathbf{R}; H^{s-2}(\mathbf{R}^{n-1})) \cap L_{\text{loc}}^{\infty}(\mathbf{R} \times \mathbf{R}^{n-1})$, by Lemma 2.3,

$$\tilde{D}^{\alpha_1} F_1(t, x') \in L^{2p_1}(\mathbf{R}^{n-1}), \quad \text{where } p_1 = \frac{s-2}{|\alpha_1|-1}$$

and

$$(2.28) \quad \|\tilde{D}^{\alpha_1} F_1(t, x')\|_{L^{2p_1}} \leq C \|\tilde{D} F_1\|_{L^\infty}^{1-\frac{1}{p_1}} \|\tilde{D} F_1\|_{H^{s-2}}^{\frac{1}{p_1}} \leq C(t).$$

for $1 \leq |\alpha_1| \leq s-1$. Similarly,

$$\psi \in C(\mathbf{R}; H^{s-2}(\mathbf{R}^{n-1})) \cap L_{\text{loc}}^\infty(\mathbf{R} \times \mathbf{R}^{n-1}),$$

with $\|\psi\|_{H^{s-2}} \leq E(t)$ and $\|\psi\|_{L^\infty} \leq C(t)$. Thus, again from Lemma 2.3,

$$\tilde{D}^{\alpha_2} \psi(t, x') \in L^{2p_2}(\mathbf{R}^{n-1}), \quad \text{where } p_2 = \frac{s-2}{|\alpha_2|},$$

and

$$(2.29) \quad \|\tilde{D}^{\alpha_2}(\psi)\|_{L^{2p_2}} \leq C \|\psi\|_{L^\infty}^{1-\frac{1}{p_2}} \|\psi\|_{H^{s-2}}^{\frac{1}{p_2}} \leq C(t) (E(t))^{\frac{1}{p_2}},$$

for $1 \leq |\alpha_2| \leq s-2$. Therefore, by Hölder's inequality,

$$(\tilde{D}^{\alpha_1} F_1(t, x')) (\tilde{D}^{\alpha_2} \psi) \in C(\mathbf{R}; L^2(\mathbf{R}^{n-1})),$$

since $((|\alpha_1|-1)+|\alpha_2|)/2(s-2) \leq 1/2$. For $|\alpha_1|=0$, $\|F_1(t, x') \tilde{D} \psi(t, x')\|_{L^2} \leq C \|F_1\|_{L^\infty} E(t)$. Therefore, from (2.28) and (2.29),

$$(2.30) \quad \|(\tilde{D}^{\alpha_1} F_1(t, x')) (\tilde{D}^{\alpha_2} \psi(t, x'))\|_{L^2} \leq C(t) E(t).$$

Since $F_2(t, x') \in C(\mathbf{R}; H^{s-2}(\mathbf{R}^{n-1})) \cap L_{\text{loc}}^\infty(\mathbf{R} \times \mathbf{R}^{n-1})$, from Lemma 2.3,

$$\tilde{D}^{\beta_1}(F_2(t, x')) \in L^{2q_1}(\mathbf{R}^{n-1}), \quad \text{where } q_1 = \frac{s-2}{|\beta_1|}$$

and

$$(2.31) \quad \|\tilde{D}^{\beta_1}(F_2(t, x'))\|_{L^{2q_1}} \leq C \|F_2\|_{L^\infty}^{1-\frac{1}{q_1}} \|F_2\|_{H^{s-2}}^{\frac{1}{q_1}} \leq C(t),$$

for $0 \leq |\beta_1| \leq s-2$. Also, again from Lemma 2.3,

$$\tilde{D}^{\beta_2} \psi(t, x') \in L^{2q_2}(\mathbf{R}^{n-1}), \quad \text{where } q_2 = \frac{s-2}{|\beta_2|},$$

and

$$(2.32) \quad \|\tilde{D}^{\beta_2} \psi\|_{L^{2q_2}} \leq C \|\psi\|_{L^\infty}^{1-\frac{1}{q_2}} \|\psi\|_{H^{s-2}}^{\frac{1}{q_2}} \leq C(t) (E(t))^{\frac{1}{q_2}},$$

for $0 \leq |\beta_2| \leq s-2$. Therefore, by Hölder's inequality,

$$\tilde{D}^{\beta_1}(F_2(t, x')) \tilde{D}^{\beta_2} \psi \in C(\mathbf{R}; L^2(\mathbf{R}^{n-1})),$$

since $(|\beta_1|+|\beta_2|)/2(s-2) \leq 1/2$. From (2.31) and (2.32),

$$(2.33) \quad \|\tilde{D}^{\beta_1}(F_2(t, x')) \tilde{D}^{\beta_2} \psi(t, x')\|_{L^2} \leq C(t) E(t).$$

Therefore, by Minkowski's inequality, Schwarz's inequality, (2.31) and (2.33),

$$\begin{aligned} |E(t) \partial_t E(t)| &\leq C \|\tilde{D} F_1\|_{L^\infty} \int \left(\varphi^{(s-2)} \right)^2 dx' + C \left(\int \left(\varphi^{(s-2)} \right)^2 dx' \right)^{\frac{1}{2}} \\ &\quad + \left[\sum_{\alpha} \|(\tilde{D}^{\alpha_1} F_1)(\tilde{D}^{\alpha_2} \psi)\|_{L^2} + \sum_{\beta} \|(\tilde{D}^{\beta_1} F_2)(\tilde{D}^{\beta_2} \psi)\|_{L^2} + \|\tilde{D}^{s-2} F_3\|_{L^2} \right] \\ &\leq C E(t) (\|\tilde{D} F_1\|_{L^\infty} E(t) + C(t) E(t) + C(t)), \end{aligned}$$

and so $|\partial_t E(t)| \leq C(t)E(t) + C(t)$. Therefore, by Gronwall's inequality, $E(t)$ is finite for all time since by assumption it is finite at $t = 0$.

3. CONORMAL SOLUTIONS FOR QUASILINEAR WAVE EQUATIONS

In this section, we will consider the conormal regularity of solutions of equation (1.1) with respect to a single characteristic hypersurface and a pair of characteristic hypersurfaces which intersect transversally.

3.1. Conormal regularity with respect to a single characteristic hypersurface.

Definition 3.1. Let Σ be a C^2 characteristic hypersurface for P_2 given in (1.1). $w(t, x) \in C(\mathbf{R}; H_{\text{loc}}^\tau(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$ is said to be *conormal with respect to Σ* if $M_1 \cdots M_j w \in C(\mathbf{R}; H_{\text{loc}}^\tau(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$ for all $C(\mathbf{R}; H_{\text{loc}}^\tau(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$ vector fields M_1, \dots, M_j which are tangent to Σ . It will be denoted by $w \in N^{\tau, \infty}(\Sigma)$. If this property holds for all $j \leq k$, then u is said to be *conormal of degree k* with respect to Σ , denoted by $w \in N^{\tau, k}(\Sigma)$.

Since surfaces $\{t = \text{constant}\}$ are space-like, it follows that we can assume without loss of generality that a characteristic hypersurface Σ for P_2 is given locally by $\{x_1 = \varphi(t, x')\}$ with nonvanishing gradient of the function $\varphi(t, x')$, and we suppose that $u \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$. Then φ satisfies (2.22), where coefficients are evaluated at $x_1 = \varphi(t, x')$ and $\varphi \in C(\mathbf{R}; H_{\text{loc}}^{s-2}(\mathbf{R}^{n-1})) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-3}(\mathbf{R}^{n-1})) \subset C^2(\mathbf{R} \times \mathbf{R}^{n-1})$, by Theorem 2.11.

Consider a change of variables:

$$(3.1) \quad y_0 = t, \quad y_1 = x_1 - \varphi(t, x'), \quad y_2 = x_2, \dots, \quad y_n = x_n,$$

with inverse transform:

$$t = y_0, \quad x_1 = y_1 + \varphi(y_0, y'), \quad x_2 = y_2, \dots, \quad x_n = y_n,$$

where $y' = (y_2, \dots, y_n) \in \mathbf{R}^{n-1}$. Under the change of variables (3.1), Σ becomes locally $S = y_1 = 0$ in the y -coordinates. We notice that $\frac{\partial}{\partial y_0}, y_1 \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_n}$ generate the tangential vector fields to S . Therefore, by the chain rule, the $C(\mathbf{R}; H_{\text{loc}}^{s-2}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-3}(\mathbf{R}^n))$ vector fields tangent to Σ are spanned by

$$(3.2) \quad \begin{aligned} M_0 &= (x_1 - \varphi(t, x')) \partial_{x_1}, \quad M_1 = \partial_t + \varphi_t(t, x') \partial_{x_1}, \\ M_j &= \partial_{x_j} + \varphi_{x_j}(t, x') \partial_{x_1}, \quad 2 \leq j \leq n. \end{aligned}$$

The idea of the proof of conormal regularity is to use commutator arguments and (1.1) to inductively establish regularity for u with respect to differentiation by the vector fields given by (3.2), for coefficients of a given regularity. The improved regularity of u , coupled with (2.22), then yields improved regularity of φ , and hence of the coefficients of the vector fields, allowing the inductive argument to be continued.

We note the following: For $2 \leq i, j \leq n$,

$$\begin{aligned}\partial_t \partial_{x_1} &= \partial_{x_1} (\partial_t + \varphi_t \partial_{x_1} - \varphi_t \partial_{x_1}) = \partial_{x_1} M_1 - \varphi_t \partial_{x_1}^2, \\ \partial_t \partial_{x_j} &= \partial_t (\partial_{x_j} + \varphi_{x_j} \partial_{x_1} - \varphi_{x_j} \partial_{x_1}) = \partial_t M_j - \varphi_{tx_j} \partial_{x_1} - \varphi_{x_j} \partial_t \partial_{x_1}, \\ \partial_{x_1} \partial_{x_j} &= \partial_{x_1} (\partial_{x_j} + \varphi_{x_j} \partial_{x_1} - \varphi_{x_j} \partial_{x_1}) = \partial_{x_1} M_j - \varphi_{x_j} \partial_{x_1}^2, \\ \partial_{x_i} \partial_{x_j} &= \partial_{x_i} (\partial_{x_j} + \varphi_{x_j} \partial_{x_1} - \varphi_{x_j} \partial_{x_1}) = \partial_{x_i} M_j - \varphi_{x_i x_j} \partial_{x_1} - \varphi_{x_j} \partial_{x_i} \partial_{x_1}.\end{aligned}$$

Let

$$F(t, x_1, x') = (\varphi_t)^2 - a_{11} + 2(a_{01} - \sum_{j=2}^n a_{0j})\varphi_t + 2 \sum_{j=2}^n a_{1j}\varphi_{x_j} - \sum_{i,j=2}^n a_{ij}\varphi_{x_i}\varphi_{x_j},$$

where

$$a_{ij} = a_{ij}(t, x_1, x', u(t, x_1, x'), Du(t, x_1, x')).$$

Since $F(t, \varphi(t, x'), x') \equiv 0$, Taylor's theorem and (2.22) imply that

$$\begin{aligned}F(t, x_1, x') &= (x_1 - \varphi) \left(\int_0^1 F_1(t, sx_1 + (1-s)\varphi, x') ds \right) \\ &\equiv (x_1 - \varphi) \tilde{F}(t, x_1, x'),\end{aligned}$$

where $F_1(t, x_1, x') = \frac{\partial F}{\partial x_1}(t, x_1, x')$. We note that by the chain rule, \tilde{F} may be considered as an integral of a smooth function of $t, x', u, Du, D^2u, \varphi$ and $D\varphi$. Therefore,

$$\begin{aligned}a_{11} &= (\varphi_t)^2 + 2(a_{01} - \sum_{j=2}^n a_{0j}\varphi_{x_j})\varphi_t + 2 \sum_{j=2}^n a_{1j}\varphi_{x_j} - \sum_{i,j=2}^n a_{ij}\varphi_{x_i}\varphi_{x_j} \\ &\quad + (x_1 - \varphi(t, x'))\tilde{F}(t, x_1, x').\end{aligned}$$

Substituting a_{11} into (1.2) and using the note above,

$$\begin{aligned}P_2(t, x, u, Du, D) &= \partial_t^2 - (\varphi_t)^2 \partial_{x_1}^2 - 2a_{01}(\partial_t \partial_{x_1} + \varphi_t \partial_{x_1}^2) \\ &\quad - 2 \sum_{j=2}^n a_{0j}(\partial_t \partial_{x_j} - \varphi_{x_j} \varphi_t \partial_{x_1}^2) - 2 \sum_{j=2}^n a_{1j}(\partial_{x_1} \partial_{x_j} + \varphi_{x_j} \partial_{x_1}^2) \\ &\quad - \sum_{i,j=2}^n a_{ij}(\partial_{x_i} \partial_{x_j} - \varphi_{x_i} \varphi_{x_j} \partial_{x_1}^2) - \tilde{F}(t, x_1, x')(x_1 - \varphi) \partial_{x_1}^2 \\ &= (\partial_t - \varphi_t \partial_{x_1}) M_1 - \varphi_{tt} \partial_{x_1} - 2a_{01} M_1 \\ &\quad - 2 \sum_{j=2}^n a_{0j}(\partial_t M_j - \varphi_{x_j} \partial_{x_1} M_1 - \varphi_{tx_j} \partial_{x_1}) - 2 \sum_{j=2}^n a_{1j} \partial_{x_1} M_j \\ &\quad - \sum_{i,j=2}^n a_{ij}(\partial_{x_j} M_j - \varphi_{x_j} \partial_{x_1} M_j - \varphi_{x_i x_j} \partial_{x_1}) \\ &\quad - \tilde{F}(t, x_1, x')(\partial_{x_1} M_0 - \partial_{x_1}).\end{aligned}$$

Therefore,

$$(3.3) \quad P_2(t, x, u, Du, D) = \sum_{i=0}^n a_i(D) M_i + b(D),$$

where $a_j(D)$ is a first order operator and the coefficients of the operator are smooth functions and integrals of smooth functions of u, Du, D^2u, φ and $D\varphi$, which will be written symbolically as $a_j(D^2u, D\varphi, D)$. Similarly, $b(D)$ is a first order operator and $b_j(D) = b(D^2u, D^2\varphi, D)$.

Since $[M_i, M_j] = 0$, $[a_i(D)M_i, M_j] = c_{ij}(D)M_i$ and $[b(D), M_j] = d_j(D)$, for $0 \leq i, j \leq n$, where

$$c_{ij}(D) = c_{ij}(D^3u, D^2\varphi, D) \quad \text{and} \quad d_j(D) = d_j(D^3u, D^3\varphi, D)$$

are first order operators, we have the following lemma.

Lemma 3.1. *Let P_2 be the second order quasilinear differential operator given in (1.2). Let Σ be a characteristic hypersurface for P_2 given locally by $\{x_1 = \varphi(t, x')\}$ with $D\varphi \neq 0$ and $\varphi \in C^3(\mathbf{R} \times \mathbf{R}^n)$. Let $u \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$. Then the $C(\mathbf{R}; H_{\text{loc}}^{s-2}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-3}(\mathbf{R}^n))$ vector fields tangent to Σ are spanned by M_0, M_1, \dots, M_n and*

$$(3.4) \quad [P_2, M_j] = \sum_{i=0}^n (c_{ij}(D)M_i + d_j(D)), \quad \text{for } j = 0, 1, \dots, n,$$

where $c_{ij}(D) = c_{ij}(D^3u, D^2\varphi, D)$ and $d_j(D) = d_j(D^3u, D^3\varphi, D)$ are first order operators.

Now, we state and prove the conormal regularity of a solution of equation (1.1) with respect to a single characteristic hypersurface.

Theorem 3.2. *Let P_2 be as above and let Σ be a characteristic hypersurface for P_2 , defined by a function with nonvanishing gradient. Suppose that $u \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$, and u satisfies the equation*

$$(3.5) \quad P_2(t, x, u, Du, D)u = f(t, x, u, Du),$$

where f is smooth. If Σ is smooth in $\{t < 0\}$ and $u \in N^{s, \infty}(\Sigma \cap \{t < 0\})$, then Σ is smooth for all time and $u \in N^{s, \infty}(\Sigma)$.

Proof. By finite propagation speed and an analysis local in time, it may be assumed that u has compact support in x for each t , and that Σ is given by $\{x_1 = \varphi(t, x')\}$. Then φ satisfies (2.22), which is a nonlinear equation of first order with $C^2(\mathbf{R}^n)$ coefficients, by Schauder's lemma, strict hyperbolicity of P_2 and Σ being a characteristic hypersurface of P_2 , because $Du \in C(\mathbf{R}; H^{s-1}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-2}(\mathbf{R}^n))$ with $s-1 > \frac{n}{2} + 3$. Therefore, $\varphi \in C^3(\mathbf{R} \times \mathbf{R}^{n-1})$. Let $v = (u, Du)$, and set $s_0 = s-2$. Then, from (2.22), the hypotheses of Theorem 2.11 are satisfied for s_0 , and therefore

$$(3.6) \quad \varphi(t, x') \in C(\mathbf{R}; H^{s_0}(\mathbf{R}^{n-1})).$$

Since the regularity of the coefficients of the vector fields M_j is *a priori* lower than that of u , it is necessary to differentiate equation (3.5) in order to use the commutator argument on an appropriate derivative of u . Let U stand for the vector of all derivatives of u up to fourth order. From (3.5) and the assumption on u , $U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. Then, by the fact that $[P_2, D] = a(D^2)$, where $a(D^2) = a(D^2u, D^2)$ is a second order operator,

$$(3.7) \quad P_2(t, x, u, Du, D)U = g(DU),$$

where g is a smooth function. The right-hand side is meaningful even if $s-5 \leq \frac{n}{2}$, because g is linear in the highest order components of DU , and the product of a function in $H^{s-4}(\mathbf{R}^n)$ with a function in the $H^{s-5}(\mathbf{R}^n)$ is in $H^{s-5}(\mathbf{R}^n)$ for $s-4 > \frac{n}{2}$. Let M stand for the vector operator $(1, M_0, \dots, M_n)$. We note that $[M_i, D] = h_i(D)$, for $i = 0, 1, \dots, n$, where $h_i(D) = h_i(D^2\varphi, D)$ is a first order operator. From (3.7) and Lemma 3.1,

$$\begin{aligned} P_2(t, x, u, Du, D)MU &= [P_2, M]U + M(g(DU)) \\ (3.8) \quad &= c(D)MU + d(D)U + (MDU)g'(DU) \\ &= P_1(D)MU + G(DU), \end{aligned}$$

where $G(DU) = G(DU, D^3\varphi)$ and $c(D) = c(D^3u, D^2\varphi, D)$, $d(D) = d(D^3u, D^3\varphi, D)$ and $P_1(D) = P_1(DU, D^2\varphi, D)$ are first order operators.

The coefficients of P_2 are in $C(\mathbf{R}; H^{s-1}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-2}(\mathbf{R}^n))$, while by (3.6) and the regularity of U , $G(DU)$ and the coefficients of $P_1(DU, D^2\varphi, D)$ are in the space $C(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. It then follows from (3.8) and Corollary 2.6 that

$$(3.9) \quad MU \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n)).$$

In order to use the improved regularity of U , it is convenient to differentiate (2.22) two times. It follows that there are smooth functions f_j such that

$$(3.10) \quad \begin{aligned} (D^2\varphi)_t &= f_1(Du(t, \varphi, x'), \tilde{D}\varphi)\tilde{D}(D^2\varphi) + f_2(D^2u(t, \varphi, x'), D^2\varphi) \\ &\quad + f_3(D^3u(t, \varphi, x'), D\varphi). \end{aligned}$$

Since $(M_1v)(t, \varphi, x') = \partial_t(v(t, \varphi, x'))$ and $(M_jv)(t, \varphi, x') = \partial_{x_j}(v(t, \varphi, x'))$ for $2 \leq j \leq n$, it follows from applying M to (3.10) that there are smooth functions F_j such that

$$(3.11) \quad \begin{aligned} (D^3\varphi)_t &= F_1(Du(t, \varphi, x'), \tilde{D}\varphi)\tilde{D}(D^3\varphi) + F_2(D^2u(t, \varphi, x'), D^2\varphi)(D^3\varphi) \\ &\quad + F_3(MD^3u(t, \varphi, x'), D^2\varphi). \end{aligned}$$

Since $DMD^3u = MD^4u + [D, M]D^3u \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n))$, we have $MD^3u \in C(\mathbf{R}; H^{s-3}(\mathbf{R}^n))$. Corollary 2.12 implies that

$$MD^3u(t, \varphi, x') \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^{n-1})).$$

Therefore, by Corollary 2.12 for $\psi = D^3\varphi$, $D^3\varphi \in C(\mathbf{R}; H^{s_0-2}(\mathbf{R}^{n-1}))$, that is, $\varphi \in C(\mathbf{R}; H^{s_0+1}(\mathbf{R}^{n-1})) \cap C^1(\mathbf{R}; H^{s_0}(\mathbf{R}^{n-1}))$.

Suppose inductively that (3.6) is improved to $\varphi \in C(\mathbf{R}; H^{s_0+k}(\mathbf{R}^{n-1}))$ and (3.9) is improved to $M^{k+1}U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. From the analogue of (3.11) for $D^{k+3}\varphi$ and Corollary 2.12 for $\psi = D^{k+3}\varphi$, it follows that

$$\varphi \in C(\mathbf{R}; H^{s_0+k+1}(\mathbf{R}^{n-1})) \cap \dots \cap C^{k+1}(\mathbf{R}; H^{s_0}(\mathbf{R}^{n-1})).$$

Then, an equation similar to (3.8) holds for $M^{k+2}U$ with coefficients depending smoothly on $DM^{k+1}U$ and $D^{k+3}\varphi$, that is,

$$P_2(t, x, u, Du, D)M^{k+2}U = P_{1(k+2)}(D)M^{k+2}U + G_{k+2}(DM^{k+1}U, D^{k+3}\varphi),$$

where $P_{1(k+2)}(D) = P_{1(k+2)}(DU, D^2\varphi, D)$ is first order operator. Consequently, (3.9) may be improved to $M^{k+2}U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. Since

an equation of the form (3.11) holds for $D^{k+4}\varphi$, with MD^3u replaced by $M^{k+2}D^3u$,

$$\begin{aligned} (D^{k+4}\varphi)_t &= F_{1(k+2)}(Du(t, \varphi, x'), \tilde{D}\varphi)\tilde{D}(D^{k+4}\varphi) \\ &\quad + F_{2(k+2)}(D^2u(t, \varphi, x'), D^2\varphi)(D^{k+4}\varphi) \\ &\quad + F_{3(k+2)}(M^{k+2}D^3u(t, \varphi, x'), D^{k+3}\varphi), \end{aligned}$$

it follows that (3.6) is then improved to $\varphi \in C(\mathbf{R}; H^{s_0+k+2}(\mathbf{R}^{n-1}))$. The induction step is complete, and the regularity of the characteristic hypersurface Σ and the conormal regularity of u are established.

3.2. Conormal regularity with respect to a pair of characteristic hypersurfaces.

Definition 3.2. Let Σ_0 and Σ_1 be C^2 characteristic hypersurfaces for P_2 given in (1.2) intersecting transversally. $w(t, x) \in C(\mathbf{R}; H_{\text{loc}}^{\tau}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$ is said to be *conormal with respect to* $\{\Sigma_0, \Sigma_1\}$ if $M_1 \cdots M_j w \in C(\mathbf{R}; H_{\text{loc}}^{\tau}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$ for all $C(\mathbf{R}; H_{\text{loc}}^{\tau}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$ vector fields M_1, \dots, M_j simultaneously tangent to both Σ_0 and Σ_1 , written $w \in N^{\tau, \infty}(\Sigma_0, \Sigma_1)$. If this property holds for all $j \leq k$, then u is said to be *conormal of degree k* with respect to $\{\Sigma_0, \Sigma_1\}$, written $w \in N^{\tau, k}(\Sigma_0, \Sigma_1)$.

Suppose that Σ_0 and Σ_1 are characteristic hypersurfaces for P_2 given locally by $\{x_1 = \varphi^0(t, x')\}$ and $\{x_1 = \varphi^1(t, x')\}$ respectively, and suppose that Σ_0 and Σ_1 intersect transversally with $\varphi_t^0 - \varphi_t^1 \neq 0$. Let $\Gamma = \Sigma_0 \cap \Sigma_1$. Suppose that $u \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$. Then φ^0 and φ^1 satisfy (2.22) with coefficients evaluated at $x_1 = \varphi^0(t, x')$ and $x_1 = \varphi^1(t, x')$ respectively, and $\varphi^0, \varphi^1 \in C(\mathbf{R}; H_{\text{loc}}^{s-2}(\mathbf{R}^{n-1})) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-3}(\mathbf{R}^{n-1})) \subset C^2(\mathbf{R} \times \mathbf{R}^{n-1})$ by Theorem 2.1 i.

Consider a change of variables:

$$(3.12) \quad y_0 = x_1 - \varphi^0(t, x'), y_1 = x_1 - \varphi^1(t, x'), y_2 = x_2, \dots, y_n = x_n.$$

We know that the Jacobian of (3.12) is $\varphi_t^1 - \varphi_t^0 \neq 0$. Let $\psi(t, x') = \varphi^0(t, x') - \varphi^1(t, x')$.

Under the change of variables (3.12), Σ_0 and Σ_1 become locally $S_0 = \{y_0 = 0\}$ and $S_1 = \{y_1 = 0\}$ respectively in the y -coordinates. The vector fields tangential to both S_0 and S_1 are generated by the vector fields $y_0 \frac{\partial}{\partial y_0}, y_1 \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_n}$. Therefore, by the chain rule, the C^2 vector fields tangent to both Σ_0 and Σ_1 are spanned by

$$\begin{aligned} M_0 &= -\frac{1}{\psi_t} (x_1 - \varphi^0(t, x')) (\partial_t + \varphi_t^1(t, x') \partial_{x_1}), \\ (3.13) \quad M_1 &= \frac{1}{\psi_t} (x_1 - \varphi^1(t, x')) (\partial_t + \varphi_t^0(t, x') \partial_{x_1}), \\ M_j &= \frac{1}{\psi_t} (-\psi_{x_j} \partial_t + (\varphi_t^0 \varphi_{x_j}^1 - \varphi_t^1 \varphi_{x_j}^0) \partial_{x_1} + \psi_t \partial_{x_j}), \quad 2 \leq j \leq n. \end{aligned}$$

Let $D_0 = \frac{1}{\psi_t} (\partial_t + \varphi_t^1 \partial_{x_1})$ and $D_1 = \frac{1}{\psi_t} (\partial_t + \varphi_t^0 \partial_{x_1})$. Then

$$M_0 = -(x_1 - \varphi^0) D_0, M_1 = (x_1 - \varphi^1) D_1 \quad \text{and} \quad M_j = -\varphi_{x_j}^0 D_0 + \varphi_{x_j}^1 D_1 + \partial_{x_j}.$$

Notice that $\partial_t = \varphi_t^0 D_0 - \varphi_t^1 D_1$, $\partial_{x_1} = D_1 - D_0$ and $\partial_{x_j} = \varphi_{x_j}^0 D_0 - \varphi_{x_j}^1 D_1 + M_j$. We have the following expressions: For $2 \leq i, j \leq n$,

$$\begin{aligned}
\partial_t^2 &= (\varphi_t^0)^2 D_0^2 - 2\varphi_t^0 \varphi_t^1 D_0 D_1 + (\varphi_t^1)^2 D_1^2 + h_{00}(D^2 \varphi, D), \\
\partial_t \partial_{x_1} &= -\varphi_t^0 D_0^2 + (\varphi_t^0 + \varphi_t^1) D_0 D_1 - \varphi_t^1 D_1^2, \\
\partial_t \partial_{x_j} &= (\varphi_t^0 \varphi_{x_j}^0) D_0^2 - (\varphi_t^1 \varphi_{x_j}^0) D_0 D_1 + (\varphi_t^1 \varphi_{x_j}^1) D_1^2 \\
&\quad + \varphi_t^0 D_0 M_j - \varphi_t^1 D_1 M_j + h_{0j}(D^2 \varphi, D), \\
\partial_{x_1}^2 &= D_0^2 - 2D_0 D_1 + D_1^2, \\
\partial_{x_1} \partial_{x_j} &= -\varphi_{x_j}^0 D_0^2 + (\varphi_{x_j}^0 + \varphi_{x_j}^1) D_0 D_1 - \varphi_{x_j}^1 D_1^2 \\
&\quad + D_1 M_j - D_0 M_j + h_{1j}(D^2 \varphi, D), \\
\partial_{x_j}^2 &= (\varphi_{x_j}^0)^2 D_0^2 - 2(\varphi_{x_j}^0 \varphi_{x_j}^1) D_0 D_1 + (\varphi_{x_j}^1)^2 D_1^2 \\
&\quad + 2\varphi_{x_j}^0 D_0 M_j - 2\varphi_{x_j}^1 D_1 M_j + M_j^2 + h_{jj}(D^2 \varphi, D), \\
\partial_{x_i} \partial_{x_j} &= (\varphi_{x_i}^0 \varphi_{x_j}^0) D_0^2 - (\varphi_{x_i}^0 \varphi_{x_j}^1 + \varphi_{x_j}^0 \varphi_{x_i}^1) D_0 D_1 + (\varphi_{x_i}^1 \varphi_{x_j}^1) D_1^2 \\
&\quad + (\varphi_{x_i}^1 + \varphi_{x_j}^0) D_0 M_j - (\varphi_{x_i}^1 + \varphi_{x_j}^1) D_1 M_j + M_i M_j + h_{ij}(D^2 \varphi, D).
\end{aligned}$$

Then

$$\begin{aligned}
P_2(t, x, u, Du, D) &= c_0 D_0^2 + c_1 D_1^2 + c_{01} D_0 D_1 + \sum_{j=2}^n c_{0j} D_0 M_j \\
(3.14) \quad &+ \sum_{j=2}^n c_{1j} D_1 M_j + \sum_{i,j=2}^n c_{ij} M_i M_j + h(D),
\end{aligned}$$

where

$$c_k = (\varphi_t^k)^2 - a_{11} + 2 \left(a_{01} - \sum_{j=2}^n a_{0j} \right) \varphi_t^k + 2 \sum_{j=2}^n a_{1j} \varphi_{x_j}^k - \sum_{i,j=2}^n a_{ij} \varphi_{x_i}^k \varphi_{x_j}^k$$

for $k = 0, 1$, $c_{ij} = c_{ij}(D^2 u, D\varphi)$ and $h(D) = h(D^2 u, D^2 \varphi, D)$ is a first order operator. Let, for $k = 0, 1$,

$$F_k(t, x_1, x') = (\varphi_t^k)^2 - a_{11} + 2 \left(a_{01} - \sum_{j=2}^n a_{0j} \right) \varphi_t^k + 2 \sum_{j=2}^n a_{1j} \varphi_{x_j}^k - \sum_{i,j=2}^n a_{ij} \varphi_{x_i}^k \varphi_{x_j}^k.$$

Since Σ_0 and Σ_1 are characteristic hypersurfaces for P_2 , $F_k(t, \varphi^k(t, x'), x') \equiv 0$. Therefore, as we see in the proof of Theorem 3.2,

$$F_k(t, x_1, x') = (x_1 - \varphi^k) \tilde{F}_k(t, x_1, x'),$$

where \tilde{F}_k is an integral of a smooth function of $t, x', u, Du, D^2 u, \varphi$ and $\tilde{D}\varphi$. Therefore, $c_0 = c_0(D^2 u, D\varphi)(x_1 - \varphi^0)$ and $c_1 = c_1(D^2 u, D\varphi)(x_1 - \varphi^1)$. By the assumption of strictly hyperbolicity, $c_{01} \neq 0$, so after division we assume

that $c_{01} \equiv 1$. Therefore,

$$(3.15) \quad \begin{aligned} P_2(t, x, u, Du, D) = & D_0 D_1 + r_{00} D_0 M_0 + r_{11} D_1 M_1 + \sum_{j=2}^n r_{0j} D_0 M_j \\ & + \sum_{j=2}^n r_{1j} D_1 M_j + \sum_{i,j=2}^n r_{ij} M_i M_j + b(D), \end{aligned}$$

where $r_{ij} = r_{ij}(D^2 u, D\varphi)$ and $b(D) = b(D^2 u, D^2 \varphi, D)$ is a first order operator.

Before we consider commutators $[P_2, M_j]$, for $j = 0, 1, \dots, n$, we will first observe several basic commutators. Let $\tilde{D}_0 = \partial_t + \varphi_t^1 \partial_{x_1}$ and $\tilde{D}_1 = \partial_t + \varphi_t^0 \partial_{x_1}$. Since we have $[\tilde{D}_0, \tilde{D}_1] = \psi_{tt} \partial_{x_1}$ and $(\tilde{D}_0(1/\psi_t)) \tilde{D}_1 - (\tilde{D}_1(1/\psi_t)) \tilde{D}_0 = -(\psi_{tt}/(\psi_t)^2) \partial_{x_1}$, $[D_0, D_1] = 0$. Therefore, we have the following lemma.

Lemma 3.3. *Let D_0 and D_1 be first order operators as above. Then*

$$D_0(x_1 - \varphi^0) = -1, D_0(x_1 - \varphi^1) = 0, D_1(x_1 - \varphi^0) = 0, \quad \text{and} \quad D_1(x_1 - \varphi^1) = 1.$$

By using Lemma 3.3, we can compute commutators $[D_j, M_j]$ for $i = 0, 1$ and $j = 0, 1, \dots, n$.

Lemma 3.4. *Let D_0, D_1 and M_j be as above. Then we have*

$$[D_0, M_0] = D_0, [D_0, M_1] = [D_1, M_0] = 0, [D_1, M_1] = D_1$$

and

$$[D_0, M_j] = [D_1, M_j] = 0 \quad \text{for } j = 2, \dots, n.$$

Proof. We consider only $[D_0, M_j]$ and $[D_1, M_j]$. We note that $[D_0, \partial_{x_j}] = [D_1, \partial_{x_j}] = (1/\psi_t) (\varphi_{ix_j}^0 D_0 - \varphi_{ix_j}^1 D_1)$, $D_0(\varphi_{x_j}^0) = D_1(\varphi_{x_j}^0) = (1/\psi_t) \varphi_{ix_j}^0$ and $D_0(\varphi_{x_j}^1) = D_1(\varphi_{x_j}^1) = (1/\psi_t) \varphi_{ix_j}^1$. Therefore, for $i = 0, 1$ and $j = 2, \dots, n$,

$$[D_i, M_j] = [D_i, -\varphi_{x_j}^0 D_0 + \varphi_{x_j}^1 D_1 + \partial_{x_j}] = 0.$$

This completes the proof of lemma.

For the commutators $[M_i, M_j]$, for $i, j = 0, 1, \dots, n$, we have the following lemma.

Lemma 3.5. *Let M_0, M_1, \dots, M_n be as above. Then, for $i, j = 2, \dots, n$ with $i \neq j$, we have*

$$[M_0, M_1] = [M_0, M_j] = [M_1, M_j] = [M_i, M_j] = 0.$$

Proof. We can easily see, from Lemma 3.3, that $[M_0, M_1] = 0$. Notice that

$$-(M_0(\varphi_{x_j}^0)) D_0 + (M_0(\varphi_{x_j}^1)) D_1 - (x_1 - \varphi^0) [D_0, \partial_{x_j}] = 0$$

and

$$-\varphi_{x_j}^0 [M_0, D_0] + (\partial_{x_j} (x_1 - \varphi^0)) D_0 = 0.$$

Similarly, we have

$$-(M_1(\varphi_{x_j}^0)) D_0 + (M_1(\varphi_{x_j}^1)) D_1 + (x_1 - \varphi^1) [D_0, \partial_{x_j}] = 0$$

and

$$\varphi_{x_j}^1 [M_1, D_1] - (\partial_{x_j} (x_1 - \varphi^1)) D_1 = 0.$$

Therefore, $[M_0, M_j] = [M_1, M_j] = 0$ for $j = 2, \dots, n$. Since

$$\begin{aligned} [M_i, M_j] &= [-\varphi_{x_i}^0 D_0 + \varphi_{x_i}^1 D_1 + \partial_{x_i}, M_j] \\ &= -\varphi_{x_i}^0 [D_0, M_j] + (M_j(\varphi_{x_i}^0)) D_0 + \varphi_{x_i}^1 [D_1, M_j] \\ &\quad - (M_j(\varphi_{x_i}^1)) D_1 + [\partial_{x_i}, M_j], \end{aligned}$$

by Lemma 3.4, it is enough to show that the coefficient of D_0 and D_1 are 0. We note that $D_0(\varphi_{x_i}^0) = D_1(\varphi_{x_i}^0) = (1/\psi_t)\varphi_{tx_i}^0$ and $D_0(\varphi_{x_i}^1) = D_1(\varphi_{x_i}^1) = (1/\psi_t)\varphi_{tx_i}^1$. From the note above, the coefficient of D_0 , which comes from the terms $(M_j(\varphi_{x_i}^0))D_0$ and $[\partial_{x_i}, M_j]$, is 0. Similarly, the coefficients of D_1 becomes 0. Therefore, $[M_i, M_j] = 0$.

Lemma 3.6. *Let D_0 , D_1 and M_j be as above. Then we have*

$$\begin{aligned} [D_0 D_1, M_0] &= D_0 D_1, \quad [D_0 D_1, M_0] = D_0 D_1, \\ [D_0 D_1, M_k] &= 0 \quad \text{for } k = 2, \dots, n, \\ [D_0 M_j, M_0] &= D_0 M_j, \quad [D_0 M_j, M_k] = 0 \quad \text{for } k \neq 0, \\ [D_1 M_j, M_1] &= D_1 M_j, \quad [D_1 M_j, M_k] = 0 \quad \text{for } k \neq 1 \end{aligned}$$

and

$$[M_i M_j, M_k] = 0 \quad \text{for } i, j, k = 0, 1, \dots, n.$$

Proof. The proof immediately follows from Lemma 3.4, Lemma 3.5 and the fact that $[AB, C] = A[B, C] + [A, C]B$.

From Lemma 3.6, we have the following lemma.

Lemma 3.7. *Let $P_2(t, x, u, Du, D)$ be a strictly hyperbolic quasilinear operator given in (1.2). Let Σ_0 and Σ_1 be characteristic hypersurfaces for P_2 intersecting transversally. Suppose that Σ_0 and Σ_1 are given locally by $\{x_1 = \varphi^0(t, x')\}$ and $\{x_1 = \varphi^1(t, x')\}$ respectively, with $\varphi^0, \varphi^1 \in C^2(\mathbb{R} \times \mathbb{R}^{n-1})$ and $\psi_t \neq 0$. Let $u \in C(\mathbb{R}; H_{\text{loc}}^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H_{\text{loc}}^{s-1}(\mathbb{R}^n))$, $s > \frac{n}{2} + 4$. Then the C^2 vector fields tangent to both Σ_0 and Σ_1 are spanned by M_0, M_1, \dots, M_n , and*

$$[P_2, M_j] = P_2 + \sum_{i=0}^n r_{ij}(D) M_i + h_j(D), \quad \text{for } j = 0, 1,$$

and

$$[P_2, M_j] = \sum_{i=0}^n r_{ij}(D) M_i + h_j(D), \quad \text{for } j = 2, \dots, n,$$

where $r_{ij}(D) = r_{ij}(D^3 u, D^2 \varphi^0, D^2 \varphi^1, D)$, $h_j(D) = h_j(D^3 u, D^3 \varphi^0, D^3 \varphi^1, D)$ are first order operators.

Proof. The proof follows from Lemma 3.16 and the fact that $[aA, B] = a[A, B] - (Ba)A$.

Now, we can state and prove the conormal regularity of a solution of equation (1.1) with respect to a pair of characteristic hypersurfaces intersecting transversally.

Theorem 3.8. Let P_2 be as above and let Σ_0 and Σ_1 be characteristic hypersurfaces for P_2 intersecting transversally in $\{t \geq 0\}$. Suppose that $u \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$, and u satisfies the equation

$$(3.16) \quad P_2(t, x, u, Du, D)u = f(t, x, u, Du),$$

where f is a smooth function of its arguments. If Σ_0 and Σ_1 are smooth in $\{t < 0\}$, and $u \in N^{s, \infty}(\Sigma_0, \Sigma_1)$ in $\{t < 0\}$, then we have:

- (1) Γ is smooth,
- (2) Σ_0 and Σ_1 are smooth outside Γ ,
- (3) u is smooth outside $\Sigma_0 \cup \Sigma_1$,
- (4) locally, $u \in N^{s, \infty}(\Sigma_0, \Sigma_1)$.

Proof. By finite propagation speed and an analysis local in time, it may be assumed that u has compact support in x for each t , and that Σ_0 and Σ_1 are given by $\{x_1 = \varphi^0(t, x')\}$ and $\{x_1 = \varphi^1(t, x')\}$ respectively. Assume that $\psi_t \neq 0$. Then φ^0 and φ^1 satisfy (2.22). From now on, as a matter of notational convenience, we use φ for both φ^0 and φ^1 . As we see in the proof of Theorem 3.2, $\varphi \in C^2(\mathbf{R} \times \mathbf{R}^{n-1})$ and for $s_0 = s - 2$,

$$(3.17) \quad \varphi(t, x') \in C(\mathbf{R}; H^{s_0}(\mathbf{R}^{n-1}))$$

since $s > \frac{n}{2} + 4$.

Let U stand for the vector of all derivatives of u up to fourth order. From (3.16) and the assumption on u , $U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$ and

$$(3.18) \quad P_2(t, x, u, Du, D)U = g(DU),$$

where the right-hand side is meaningful as we noted in the proof of Theorem 3.2. Let M stand for the vector operator $(1, M_0, M_1, \dots, M_n)$. We note that $[M_i, D] = h_i(D^2\varphi, D)$. From (3.18) and Lemma 3.7,

$$(3.19) \quad \begin{aligned} P_2(t, x, u, Du, D)MU &= [P_2, M]U + M(g(DU)) \\ &= P_2U + c(D)MU + d(D)U + (MDU)g'(DU) \\ &= P_1(D)MU + G(DU), \end{aligned}$$

where $G(DU) = G(DU, D^3\varphi)$ and $c(D) = c(D^3u, D^2\varphi, D)$, $d(D) = d(D^3u, D^3\varphi, D)$ and $P_1(D) = P_1(DU, D^2\varphi, D)$ are first order operators.

The coefficients of P_2 are in $C(\mathbf{R}; H^{s-1}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-2}(\mathbf{R}^n))$, while by (3.17) and (2.22), $G(DU)$ and the coefficients of $P_1(DU, D^2\varphi, D)$ are in the space $C(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. It then follows from (3.19) and Corollary 2.6 that

$$(3.20) \quad MU \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n)).$$

In order to use the improved regularity of U , it is convenient to differentiate (2.22) two times. It follows that there are smooth functions f_j such that

$$(3.21) \quad \begin{aligned} (D^2\varphi)_t &= f_1(Du(t, \varphi, x'), \tilde{D}\varphi)\tilde{D}(D^2\varphi) + f_2(D^2u(t, \varphi, x'), D^2\varphi) \\ &\quad + f_3(D^3u(t, \varphi, x'), D\varphi), \end{aligned}$$

where \tilde{D} is the vector operator $(\partial_{x_2}, \dots, \partial_{x_n})$.

Let $\phi_j = \varphi_t^0 \varphi_{x_j}^1 - \varphi_t^1 \varphi_{x_j}^0$, $\mathcal{L}^0 = L_1^0, L_j^0$ and $\mathcal{L}^1 = L_1^1, L_j^1$, where $L_1^0 = (\psi/\psi_t)(\partial_t + \varphi_t^0 \partial_{x_1})$, $L_1^1 = (\psi/\psi_t)(\partial_t + \varphi_t^1 \partial_{x_1})$ and

$$L_j^0 = L_j^1 = (1/\psi_t)(-\psi_{x_j} \partial_t + \phi_j \partial_{x_1} + \psi_t \partial_{x_j}).$$

We note that $[L, D] = k(D^2\psi, \partial_t, \partial_{x_j}) + k(D^2\psi, D^2\varphi, \partial_{x_1})$ for $L \in \mathcal{L}^0 \cup \mathcal{L}^1$. Since $\partial_t = (\psi_t/\psi)\tilde{L}_1$ and $\partial_{x_j} = (\psi_{x_j}/\psi)\tilde{L}_1 - \tilde{L}_j$ away from $\psi = 0$, where $\tilde{L}_1 = (\psi/\psi_t)\partial_t$ and $\tilde{L}_j = (1/\psi_t)(-\psi_{x_j}\partial_t + \psi_t\partial_{x_j})$, $D(D^2\varphi) = l(D\psi)(\tilde{L}D^2\varphi)$ away from $\{\psi = 0\}$.

Since $(Lv)(t, \varphi^i, x') = L(v(t, \varphi^i, x'))$ for $L \in \mathcal{L}^i$, $i = 0, 1$, it follows from applying $L \in \mathcal{L}^i$ to (3.21) with φ^i in place of φ respectively, that there are smooth functions F_j away from $\psi = 0$ such that

$$(3.22) \quad \begin{aligned} (\tilde{L}D^2\varphi^i)_t &= F_1(Du(t, \varphi^i, x'), \tilde{D}\varphi^i)\tilde{D}(\tilde{L}D^2\varphi^i) \\ &\quad + F_2(D^2u(t, \varphi^i, x'), D^2\varphi^i, D^2\psi)(\tilde{L}D^2\varphi^i) \\ &\quad + F_3(LD^3u(t, \varphi^i, x'), D^2\varphi^i, D\psi). \end{aligned}$$

Since $DMD^3u = MD^4u + [D, M]D^3u \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n))$, we have $MD^3u \in C(\mathbf{R}; H^{s-3}(\mathbf{R}^n))$. By Corollary 2.12, $LD^3u(t, \varphi^i, x') \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^{n-1}))$. Therefore, Corollary 2.12 implies that $\tilde{L}D^2\varphi \in C(\mathbf{R}; H^{s_0-2}(\mathbf{R}^{n-1}))$.

The functions φ^i (and hence ψ) are not necessarily smooth across $\{\varphi^0 = \varphi^1\}$ (that is, $\{\psi = 0\}$), as is known from the example of Messer [9]. On the other hand, Γ itself (defined by $\{\psi = 0\} \cap \{x_1 = \varphi^0\}$) will be seen to be smooth. In order to avoid the functions φ^i on the set $\{\psi = 0\} \cap \{x_1 = \varphi^0\}$, we use finite propagation speed and analyze regularity separately in the past and in the future.

Since $\psi_t \neq 0$, by the implicit function theorem, there is a function $t = \tau(x')$ such that $\psi(\tau(x'), x') \equiv 0$. Therefore, Γ can be expressed by $\{(t, x_1, x') : \psi(\tau(x'), x') = 0\} \cap \{(t, x_1, x') : x_1 = \varphi^0(\tau(x'), x')\}$. For sufficiently small $\epsilon > 0$, let

$$\begin{aligned} R_1 &= \{(t, x_1, x') : t + \epsilon|x_1 - \varphi^0(\tau(x'), x')|^2 < \tau(x')\}, \\ R_2 &= \{(t, x_1, x') : t + \epsilon|x_1 - \varphi^0(\tau(x'), x')|^2 > \tau(x')\}, \\ R_3 &= \mathbf{R}^{n+1} \setminus (R_1 \cup R_2). \end{aligned}$$

On R_3 , u is smooth away from Γ by finite propagation speed. Therefore, we will consider the lower region R_1 and upper region R_2 . Let

$$M_j^{R_i} = \begin{cases} M_j & \text{on } R_i \text{ for } i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have Lemma 3.7, (3.19), (3.20) and (3.22) for $M_j^{R_i}$. Applying M^{R_i} to equation (3.19), we obtain the equation

$$P_2(M^{R_i})^2U = P_{12}(D)(M^{R_i})^2U + G_2(DM^{R_i}U, M^{R_i}D^2\varphi).$$

From (3.22), $M^{R_i}D^2\varphi \in C(\mathbf{R}; H^{s_0-2}(\mathbf{R}^{n-1}))$ and then, by Corollary 2.12,

$$(3.23) \quad (M^{R_i})^2U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n)).$$

Applying $L \in \mathcal{L}^i$ to (3.22), we have the equation

$$(3.24) \quad \begin{aligned} (\tilde{L}^2D^2\varphi^i)_t &= F_{12}(Du(t, \varphi^i, x'), \tilde{D}\varphi^i)\tilde{D}(\tilde{L}^2D^2\varphi^i) \\ &\quad + F_{22}(LD^2u(t, \varphi^i, x'), \tilde{L}D^2\varphi^i, D^2\psi)(\tilde{L}^2D^2\varphi^i) \\ &\quad + F_{32}(L^2D^3u(t, \varphi^i, x'), \tilde{L}D^2\varphi^i, \tilde{L}D^2\psi). \end{aligned}$$

A similar argument as above implies that $\tilde{L}^2 D^2 \varphi \in C(\mathbf{R}; H^{s_0-2}(\mathbf{R}^{n-1}))$.

By differentiating $\psi(\tau(x'), x') \equiv 0$ with respect to x_j , we have

$$(3.25) \quad \psi_t(\tau(x'), x') \frac{\partial \tau}{\partial x_j} + \psi_{x_j}(\tau(x'), x') = 0 \quad \text{or} \quad \frac{\partial \tau}{\partial x_j} = -\frac{\psi_{x_j}(\tau(x'), x')}{\psi_t(\tau(x'), x')}.$$

Therefore, the regularity of Γ depends only on the regularity of φ since $\tau(x') \in C^2(\mathbf{R}^{n-1})$ and $\psi = \varphi^0 - \varphi^1$. More precisely, we have the following:

On Γ , the vector fields M_2, \dots, M_n are equivalent to the vector fields $\partial_{x_2}, \dots, \partial_{x_n}$ and, from (3.25),

$$(3.26) \quad \begin{aligned} \partial_{x_j}(\varphi(\tau(x'), x')) &= \frac{\partial \varphi}{\partial t}(\tau(x'), x') \frac{\partial \tau}{\partial x_j} + \frac{\partial \varphi}{\partial x_j}(\tau(x'), x') \\ &= -\frac{\psi_{x_j}}{\psi_t}(\tau(x'), x') \partial_t \varphi(\tau(x'), x') + \partial_{x_j} \varphi(\tau(x'), x') \\ &= (M_j \varphi)(\tau(x'), x'). \end{aligned}$$

Therefore, the improved regularity of φ with respect to M_2, \dots, M_n implies the improved regularity of $\varphi(\tau(x'), x')$ with respect to $\partial_2, \dots, \partial_n$ and of $\tau(x')$ with respect to $\partial_2, \dots, \partial_n$ since we can write (3.25) as

$$(3.27) \quad \frac{\partial \tau}{\partial x_j} = -\frac{\psi_{x_j}(\tau(x'), x')}{\psi_t(\tau(x'), x')} = F\left(\varphi_{x_j}(\tau(x'), x'), \varphi_t(\tau(x'), x')\right),$$

where F is a smooth function of its arguments. Moreover, from (3.26),

$$\begin{aligned} \frac{\partial^2 \tau}{\partial x_i \partial x_j} &= \frac{1}{\psi_t^2} \left[\left(-\psi_{tx_j} \frac{\partial \tau}{\partial x_i} - \psi_{x_i x_j} \right) \psi_t + \psi_{x_j} \left(\psi_{tt} \frac{\partial \tau}{\partial x_i} + \psi_{tx_i} \right) \right] \\ &= \frac{1}{\psi_t^3} \left(\psi_{tx_j} \psi_{x_j} \psi_t - \psi_{x_i x_j} \psi_t \psi_t - \psi_{x_j} \psi_{tt} \psi_{x_i} + \psi_{tx_i} \psi_t \psi_{x_j} \right), \end{aligned}$$

evaluated at $(\tau(x'), x')$, and

$$\begin{aligned} &\partial_{x_i} \left(\partial_{x_j}(\varphi(\tau(x'), x')) \right) \\ &= \left(\varphi_{tt} \frac{\partial \tau}{\partial x_i} + \varphi_{tx_i} \right) \frac{\partial \tau}{\partial x_j} + \varphi_t \frac{\partial^2 \tau}{\partial x_i \partial x_j} + \varphi_{tx_j} \frac{\partial \tau}{\partial x_i} + \varphi_{x_i x_j} \\ &= (M_i M_j \varphi)(\tau(x'), x'). \end{aligned}$$

Suppose inductively that (3.17) is improved to $\tilde{L}^k D^2 \varphi^i \in C(\mathbf{R}; H^{s_0-2}(\mathbf{R}^{n-1}))$ and (3.23) is improved to $(M^{R_i})^{k+1} U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. From the analogue of (3.24) for $\tilde{L}^{k+1} D^2 \varphi^i$ and Corollary 2.12, it follows that, on each R_i ,

$$\varphi \in C(\mathbf{R}; H^{s_0+k+1}(\mathbf{R}^{n-1})) \cap \dots \cap C^{k+1}(\mathbf{R}; H^{s_0}(\mathbf{R}^{n-1})),$$

since $\partial_t = (\psi_t/\psi)\tilde{L}_1$ and $\partial_{x_j} = (\psi_{x_j}/\psi)\tilde{L}_1 - \tilde{L}_j$. Then, an equation similar to (3.19) holds for $(M^{R_i})^{k+2} U$ with coefficients depending smoothly on $D(M^{R_i})^{k+1} U$ and $(M^{R_i})^{k+1} D^2 \varphi$, that is,

$$\begin{aligned} &P_2(t, x, u, Du, D)(M^{R_i})^{k+2} U \\ &= P_{1(k+2)}(D)(M^{R_i})^{k+2} U + G_{k+2}(D(M^{R_i})^{k+1} U, (M^{R_i})^{k+1} D^2 \varphi), \end{aligned}$$

where $P_{1(k+2)}(D) = P_{1(k+2)}(DU, D^2\varphi, D)$ is a first order operator. Consequently, (3.20) may be improved to $(M^{R_i})^{k+2}U \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H^{s-5}(\mathbf{R}^n))$. Since an equation of the form (3.24) holds for $\tilde{L}^{k+2}D^2\varphi^i$, that is,

$$\begin{aligned} (\tilde{L}^{k+2}D^2\varphi^i)_t &= F_{1(k+2)}(Du(t, \varphi^i, x'), \tilde{D}\varphi)\tilde{D}(\tilde{L}^{k+2}D^2\varphi^i) \\ &\quad + F_{2(k+2)}(L^{k+1}Du(t, \varphi^i, x'), \tilde{L}D\varphi^i, D^2\psi)(\tilde{L}^{k+2}D^2\varphi^i) \\ &\quad + F_{3(k+2)}(L^{k+2}D^3u(t, \varphi^i, x'), \tilde{L}^{k+1}D^2\varphi^i, \tilde{L}^{k+1}D^2\psi), \end{aligned}$$

it follows that (3.17) is then improved to $\varphi \in C(\mathbf{R}; H^{s_0+k+2}(\mathbf{R}^{n-1}))$ on each R_i . At the same time, the regularity of Γ is improved along with the regularity of φ by (3.27). The induction step is complete. Therefore, the regularity of Γ and the characteristic hypersurfaces Σ_0, Σ_1 away from Γ , and the conormal regularity of u are established.

Lemma 3.9. *Let P_2 be as above and $v(t, x)$ satisfy the equation $P_2v(t, x) = g(Dv)$, where g is a smooth function of its arguments. Let, for all k_1, \dots, k_n and for $s > \frac{n}{2} + 4$,*

$$\begin{aligned} (x_1\partial_{x_1})^{k_1}(\partial_{x_2})^{k_2}\dots(\partial_{x_n})^{k_n}v(0, x) &\in H^{s-4}(\mathbf{R}^n), \\ (x_1\partial_{x_1})^{k_1}(\partial_{x_2})^{k_2}\dots(\partial_{x_n})^{k_n}v_t(0, x) &\in H^{s-5}(\mathbf{R}^n). \end{aligned}$$

Then, for any $k > 0$, $x_1^k\partial_t^k v(0, x)$, $x_1^k\partial_{x_1}^k v(0, x)$, $\sum_{l_1+l_2=k} x_1^k\partial_{x_1}^{l_1}\partial_{x_2}^{l_2}v(0, x)$ and $\sum_{l_1+l_2=k} x_1^k\partial_{x_1}^{l_1}\partial_{x_2}^{l_2}v_t(0, x)$, where $l_1 > 0$ and $j > 0$, are in the space $H^{s-4}(\mathbf{R}^n)$.

Proof. We prove this by induction on k . For $k = 1$, $x_1\partial_t v(0, x) \in H^{s-4}(\mathbf{R}^n)$ since $\partial_{x_1}(x_1\partial_t)v(0, x) = \partial_tv(0, x) + (x_1\partial_{x_1})v_t(0, x)$ and $\partial_{x_j}(x_1\partial_t)v(0, x) = x_1\partial_{x_j}v_t(0, x)$ are in the space $H^{s-5}(\mathbf{R}^n)$. Also $x_1\partial_{x_j}v(0, x) \in H^{s-4}(\mathbf{R}^n)$ by the hypothesis. Suppose that the lemma is true for $k - 1$. Note that $(x_1\partial_{x_1})^k$ is a linear combination of $x_1\partial_{x_1}, \dots, x_1^k\partial_{x_1}^k$. Therefore $x_1^k\partial_{x_1}^k$ is a linear combination of $(x_1\partial_{x_1}), \dots, (x_1\partial_{x_1})^k$ and so $x_1^k\partial_{x_1}^k v(0, x) \in H^{s-4}(\mathbf{R}^n)$ and $x_1^k\partial_{x_1}^l v_t(0, x) \in H^{s-5}(\mathbf{R}^n)$, for $1 \leq l \leq k$, by the induction hypothesis.

For $l_1 = 1$, we want to examine the regularities of $(x_1^k\partial_t\partial_{x_1}^{k-1})v(0, x)$ and $x_1^k\partial_t\partial_{x_1}^{k-1}v(0, x)$. Since $\partial_{x_1}(x_1^k\partial_t\partial_{x_1}^{k-1})v(0, x) = kx_1^{k-1}\partial_{x_1}^{k-1}v_t(0, x) + x_1^k\partial_{x_1}^k v_t(0, x)$ and $\partial_{x_j}(x_1^k\partial_t\partial_{x_1}^{k-1})v(0, x) = x_1(x_1^{k-1}\partial_{x_1}^{k-1})(\partial_{x_j})v_t(0, x)$ for $j > 1$, $(x_1^k\partial_t\partial_{x_1}^{k-1})v(0, x) \in H^{s-4}(\mathbf{R}^n)$. Similarly $(x_1^k\partial_t\partial_{x_j}^{k-1})v(0, x)$ is in the space $H^{s-4}(\mathbf{R}^n)$ for $j > 1$. For $l_1 > 1$, $\partial_t^2 = P_2 + \sum a_{ij}(Du)\partial_{x_i}\partial_{x_j}$ from (1.1) and so

$$\begin{aligned} x_1^k\partial_t^k v(0, x) &= x_1^k\partial_t^2(\partial_t^{k-2}v)(0, x) \\ &= x_1^k P_2 V(0, x) + x_1^k \sum a_{ij}(Du)(\partial_{x_i}\partial_{x_j} V)(0, x) \\ &= x_1^k \tilde{g}(DV)(0, x) + x_1^k \sum a_{ij}(Du)(\partial_{x_i}\partial_{x_j} V)(0, x), \end{aligned}$$

where $V(t, x) = \partial_t^{k-2}v(t, x)$ and $P_2V = \tilde{g}(DV)$ given by differentiating $P_2v = g(Dv)$ by ∂_t^{k-2} . Therefore, by the induction hypothesis and Schauder's lemma, $x_1^k \partial_t^k v(0, x) \in H^{s-4}(\mathbf{R}^n)$. Similarly,

$$x_1^k \partial_t^{l_1} \partial_{x_1}^{l_2} v(0, x) = x_1^k \partial_{x_1}^{l_2} \partial_t^2 (\partial_t^{l_1-2} v)(0, x)$$

and so $x_1^k \partial_t^{l_1} \partial_{x_1}^{l_2} v(0, x) \in H^{s-4}(\mathbf{R}^n)$. This completes the proof of lemma.

Theorem 3.10. *Let P_2 be as above and let Σ_0 and Σ_1 be characteristic hypersurfaces for P_2 in $t > 0$ intersecting transversally in $t = 0$, with $\Gamma = \Sigma_0 \cap \Sigma_1$. Suppose that Σ_0 and Σ_1 are given locally by $x_1 = \varphi^0(t, x')$ and $x_1 = \varphi^1(t, x')$ respectively with $\psi_t \neq 0$, where $\psi = \varphi^0 - \varphi^1$. Suppose that $u \in C(\mathbf{R}^+; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}^+; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$, and u satisfies the equation*

$$P_2(t, x, u, Du, D)u = f(t, x, u, Du),$$

where f is a smooth function of its arguments. Suppose that Γ is smooth. If $u(0, x)$ and $u_t(0, x)$ belong to $N^{s, \infty}(\Gamma)$, then we have:

- (1) Σ_0 and Σ_1 are smooth,
- (2) u is smooth outside $\Sigma_0 \cup \Sigma_1$,
- (3) locally, $u \in N^{s, \infty}(\Sigma_0, \Sigma_1)$.

Proof. By finite propagation speed and an analysis local in time, it may be assumed that u has compact support in x for each t . From now on, as a matter of notational convenience, we use φ for both φ^0 and φ^1 . As we see in the proof of Theorem 3.2, $\varphi \in C^2(\mathbf{R}^+ \times \mathbf{R}^{n-1})$ and for $s_0 = s - 2$,

$$(3.28) \quad \varphi(t, x') \in C(\mathbf{R}^+; H^{s_0}(\mathbf{R}^{n-1}))$$

since $s > \frac{n}{2} + 4$.

Let U stand for the vector of all derivatives of u up to fourth order. By a smooth change of variables, we may assume that $\Gamma = \{x_1 = 0\}$. Then the vector fields tangential to Γ are generated by $x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$.

Let M stand for the vector operator $(1, M_0, M_1, \dots, M_n)$. We note that $[M_i, D] = h_i(D^2\varphi, D)$. From (3.19),

$$P_2(t, x, u, Du, D)MU = P_1(D)MU + G(DU),$$

where $G(DU) = G(DU, D^3\varphi)$ and $c(D) = c(D^3u, D^2\varphi, D)$, $d(D) = d(D^3u, D^3\varphi, D)$ and $P_1(D) = P_1(DU, D^2\varphi, D)$ are first order operators.

The coefficients of P_2 are in $C(\mathbf{R}^+; H^{s-1}(\mathbf{R}^n)) \cap C^1(\mathbf{R}^+; H^{s-2}(\mathbf{R}^n))$, while by (3.28) and (2.22), $G(DU)$ and the coefficients of $P_1(DU, D^2\varphi, D)$ are in the space $C(\mathbf{R}^+; H^{s-5}(\mathbf{R}^n))$.

Since $u(0, x)$ and $u_t(0, x)$ are conormal with respect to Γ , we have

$$\begin{aligned} (x_1 \partial_{x_1})^{k_1} (\partial_{x_2})^{k_2} \dots (\partial_{x_n})^{k_n} U(0, x) &\in H^{s-4}(\mathbf{R}^n), \\ (x_1 \partial_{x_1})^{k_1} (\partial_{x_2})^{k_2} \dots (\partial_{x_n})^{k_n} U_t(0, x) &\in H^{s-5}(\mathbf{R}^n). \end{aligned}$$

Since $\varphi^0(0, x') \equiv 0 \equiv \varphi^1(0, x')$, by Taylor's Theorem, for $i = 0$ and 1 ,

$$\varphi^i(0, x) = t \int_0^1 \varphi_1^i(st, x') ds \equiv t \tilde{\varphi}^i(t, x'),$$

where $\varphi_1^i(t, x') = \frac{\partial \varphi^i}{\partial t}(t, x')$.

Let $\tilde{\psi} = \tilde{\varphi}^0 - \tilde{\varphi}^1$. Then $\psi = t\tilde{\psi}$. We note that $\varphi_t^i = \tilde{\varphi}^i + t\tilde{\varphi}_t^i$, $\varphi_{x_j}^i = t\tilde{\varphi}_{x_j}^i$, $\psi_{x_j} = t\tilde{\psi}_{x_j}$ and

$$\phi_j = \varphi_t^0 \varphi_{x_j}^1 - \varphi_t^1 \varphi_{x_j}^0 = t \left(\tilde{\varphi}^0 \tilde{\varphi}_{x_j}^1 - t\tilde{\varphi}_t^0 \tilde{\varphi}_{x_j}^1 - \tilde{\varphi}^1 \tilde{\varphi}_{x_j}^0 - t\tilde{\varphi}_t^1 \tilde{\varphi}_{x_j}^0 \right).$$

Therefore, $\phi_j(0, x') \equiv 0$, and $\psi_{x_j}(0, x') \equiv 0$.

We want to examine the regularity of $MU(0, x)$ by using the conormal regularity of the initial data. First, from (3.13)

$$\begin{aligned} (M_0 U)(0, x) &= - \left(x_1 - \varphi^0(0, x') \right) \frac{1}{\psi_t}(0, x') \left(\partial_t + \varphi_t^1(0, x') \partial_{x_1} \right) U(0, x) \\ &= - \frac{1}{\psi_t}(0, x') (x_1 \partial_t) U(0, x) - \varphi_t^1(0, x') \left(x_1 \partial_{x_1} \right) U(0, x). \end{aligned}$$

Since $\partial_{x_1}(x_1 \partial_t) U(0, x) = U_t(0, x) + (x_1 \partial_{x_1}) U_t(0, x) \in H^{s-5}(\mathbf{R}^n)$ and $\partial_{x_j}(x_1 \partial_t) U(0, x) = (x_1 \partial_{x_j}) U_t(0, x) \in H^{s-5}(\mathbf{R}^n)$, $(x_1 \partial_t) U(0, x) \in H^{s-4}(\mathbf{R}^n)$ and so $(M_0 U)(0, x) \in H^{s-4}(\mathbf{R}^n)$.

Second,

$$\begin{aligned} (M_1 U)(0, x) &= \left(x_1 - \varphi^1(0, x') \right) \frac{1}{\psi_t}(0, x') \left(\partial_t + \varphi_t^0(0, x') \partial_{x_1} \right) U(0, x) \\ &= \frac{1}{\psi_t}(0, x') (x_1 \partial_t) U(0, x) + \frac{1}{\psi_t}(0, x') \varphi_t^0(0, x') \left(x_1 \partial_{x_1} \right) U(0, x). \end{aligned}$$

Therefore $(M_1 U)(0, x) \in H^{s-4}(\mathbf{R}^n)$. Finally, for $j = 2, \dots, n$,

$$\begin{aligned} (M_j U)(0, x) &= \frac{1}{\psi_t}(0, x') \left(-\psi_j(0, x') \partial_t U(0, x) + \phi_j \partial_{x_1} U(0, x) + \psi_t \partial_{x_j} \right) U(0, x) \\ &= \partial_{x_j} U(0, x), \end{aligned}$$

and so $(M_j U)(0, x) \in H^{s-4}(\mathbf{R}^n)$.

It then follows from Corollary 2.6 that

$$MU \in C(\mathbf{R}^+; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}^+; H^{s-5}(\mathbf{R}^n)).$$

In order to use the improved regularity of U , it is convenient to differentiate (2.22) two times and then we get (3.21).

Let \mathcal{L}^0 and \mathcal{L}^1 be as in the proof of Theorem 3.8. As we see in the proof of Theorem 3.8, it follows from (3.22) and Corollary 2.12 that $\tilde{L}D^2\varphi^i \in C(\mathbf{R}^+; H^{s_0-2}(\mathbf{R}^{n-1}))$ since $\tilde{L}D^2\varphi^i(0, x') \in H^{s_0-2}(\mathbf{R}^{n-1})$.

Applying M to equation (3.19), we obtain the equation

$$P_2(M^2 U) = P_{12}(D)(M^2 U) + G_2(DMU, MD^2\varphi),$$

with $P_{12} = P_{12}(DU, D^2\varphi)$ a first order operator.

From (3.22), $MD^2\varphi \in C(\mathbf{R}^+; H^{s_0-2}(\mathbf{R}^{n-1}))$ and then, by Corollary 2.12 and Lemma 3.9,

$$(3.29) \quad M^2 U(t, x) \in C(\mathbf{R}^+; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}^+; H^{s-5}(\mathbf{R}^n)).$$

Applying $L \in \mathcal{L}^i$ to (3.22), we have the equation

$$(3.30) \quad \begin{aligned} (\tilde{L}^2 D^2 \varphi^i)_t &= F_{12}(Du(t, \varphi^i, x'), \tilde{D}\varphi^i) \tilde{D}(\tilde{L}^2 D^2 \varphi^i) \\ &+ F_{22}(LD^2 u(t, \varphi^i, x'), \tilde{L} D^2 \varphi^i, D^2 \psi)(\tilde{L}^2 D^2 \varphi^i) \\ &+ F_{32}(L^2 D^3 u(t, \varphi^i, x'), \tilde{L} D^2 \varphi^i, \tilde{L} D^2 \psi). \end{aligned}$$

A similar argument as above implies that $\tilde{L}^2 D^2 \varphi \in C(\mathbf{R}^+; H^{s_0-2}(\mathbf{R}^{n-1}))$.

Suppose inductively that equation (3.28) is improved to $\tilde{L}^k D^2 \varphi^i \in C(\mathbf{R}^+; H^{s_0-2}(\mathbf{R}^{n-1}))$ and (3.29) is improved to $M^{k+1} U \in C(\mathbf{R}^+; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}^+; H^{s-5}(\mathbf{R}^n))$. From the analogue of (3.30) for $\tilde{L}^{k+1} D^2 \varphi^i$ and Corollary 2.12, it follows that

$$\varphi \in C(\mathbf{R}^+; H^{s_0+k+1}(\mathbf{R}^{n-1})) \cap \dots \cap C^{k+1}(\mathbf{R}^+; H^{s_0}(\mathbf{R}^{n-1})),$$

since $\partial_t = (\psi_t/\psi)\tilde{L}_1$ and $\partial_{x_j} = (\psi_{x_j}/\psi)\tilde{L}_1 - \tilde{L}_j$. Then, an equation similar to (3.19) holds for $M^{k+2} U$ with coefficients depending smoothly on $DM^{k+1} U$ and $M^{k+1} D^2 \varphi$, that is,

$$\begin{aligned} P_2(t, x, u, Du, D)M^{k+2} U \\ = P_{1(k+2)}(D)M^{k+2} U + G_{k+2}(DM^{k+1} U, M^{k+1} D^2 \varphi), \end{aligned}$$

where $P_{1(k+2)}(D) = P_{1(k+2)}(DU, D^2 \varphi, D)$ is a first order operator. Consequently, (3.20) may be improved to

$$M^{k+2} U \in C(\mathbf{R}^+; H^{s-4}(\mathbf{R}^n)) \cap C^1(\mathbf{R}^+; H^{s-5}(\mathbf{R}^n)).$$

Since an equation of the form (3.31) holds for $\tilde{L}^{k+2} D^2 \varphi^i$, that is,

$$\begin{aligned} (\tilde{L}^{k+2} D^2 \varphi^i)_t &= F_{1(k+2)}(Du(t, \varphi^i, x'), \tilde{D}\varphi) \tilde{D}(\tilde{L}^{k+2} D^2 \varphi^i) \\ &+ F_{2(k+2)}(L^{k+1} Du(t, \varphi^i, x'), \tilde{L} D^2 \varphi^i, D^2 \psi)(\tilde{L}^{k+2} D^2 \varphi^i) \\ &+ F_{3(k+2)}(L^{k+2} D^3 u(t, \varphi^i, x'), \tilde{L}^{k+1} D^2 \varphi^i, \tilde{L}^{k+1} D^2 \psi), \end{aligned}$$

it follows that (3.28) is then improved to $\varphi \in C(\mathbf{R}^+; H^{s_0+k+2}(\mathbf{R}^{n-1}))$. The induction step is complete. Therefore, the conormal regularity of u is established.

4. PIECEWISE SMOOTHNESS

In this section, we will consider the piecewise smoothness of a solution u of the equation $P_2(t, x, u, Du, D)u = f(Du)$.

Definition 4.1. Suppose that an open set Ω in \mathbf{R}^{n+1} is divided into finitely many open components, Ω_i , by a family of $C(\mathbf{R}; H_{\text{loc}}^\tau(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{\tau-1}(\mathbf{R}^n))$, $\tau > \frac{n}{2}$, hypersurfaces H_1, \dots, H_N . A function $u(t, x)$ is called piecewise continuous if its restriction to each Ω_i has a continuous extension to $\overline{\Omega}_i$, written

$$u \in C(\Omega \setminus H_1, \dots, H_N).$$

When the H_i are clear from the context we will just say that u is piecewise continuous (p.c.). A function $u(t, x)$ is called piecewise smooth if $D^\alpha u$ is piecewise continuous for all α , written

$$u \in C^\infty(\Omega \setminus H_1, \dots, H_N).$$

In this section, by u being piecewise smooth we mean that $D^\alpha u$ is piecewise continuous for all $|\alpha| \geq k$, for some k . Let \mathcal{O} be an open neighborhood of the origin and let $\mathcal{O}_0 = \mathcal{O} \cap \{t = 0\}$. Let Σ_0 , Σ_1 and Γ be as in section 3.2. With a slightly changed time coordinate, we may assume, by strict hyperbolicity of P_2 and the hypothesis of Theorem 3.8 since Γ is smooth, that Γ divides \mathcal{O}_0 into two connected components and $\Sigma_0 \cup \Sigma_1$ divides \mathcal{O} in four connected components.

In order to treat piecewise smoothness, we treat the initial value problem, as in Rauch-Reed [15], instead of making assumptions in $\{t < 0\}$.

Theorem 4.1. *Let P_2 be as in (1.2) and let Σ_0 and Σ_1 be characteristic hypersurfaces for P_2 intersecting transversally in $\{t = 0\}$ with $\Gamma = \Sigma_0 \cap \Sigma_1$. Suppose that $u \in C(\mathbf{R}; H_{\text{loc}}^s(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-1}(\mathbf{R}^n))$, $s > \frac{n}{2} + 4$, and u satisfies the equation*

$$(4.1) \quad P_2(t, x, u, Du, D)u = f(t, x, u, Du),$$

where f is a smooth function of its arguments. Suppose that Γ is smooth. Suppose that $u(0, x) \in H_{\text{loc}}^s(\mathbf{R}^n)$ and $u_t(0, x) \in H_{\text{loc}}^{s-1}(\mathbf{R}^n)$ and $u(0, x), u_t(0, x) \in C^\infty(\mathcal{O}_0 \setminus \Gamma)$. Then $u(t, x) \in C^\infty(\mathcal{O} \setminus \Sigma_0, \Sigma_1)$.

Proof. We remark that the same argument as in section 3.2, except for the initial value problem instead of the case in which the hypotheses are made in $\{t < 0\}$, implies that Σ_0 and Σ_1 are smooth away from Γ and $u \in N^{s, \infty}(\Sigma_0, \Sigma_1)$. Therefore u is smooth outside $\Sigma_0 \cup \Sigma_1$.

It suffices to prove the theorem locally. We use φ , D_0 , D_1 and M_j as in the proof of Theorem 3.8. We will show for each $m \geq 0$, $i + j = m$ and for all α , that

$$(4.2) \quad D_0^i D_1^j M^\alpha v, D_0^i D_1^j M^\alpha w \in p.c.,$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$, $v = D_0(D^3 u)$, $w = D_1(D^3 u)$ and M is the vector operator (M_0, M_1, \dots, M_n) . This will imply that the functions φ^0 and φ^1 are themselves piecewise smooth and hence the vector fields have piecewise smooth coefficients. Since D_0 , D_1 and $\tilde{M} = (M_2, \dots, M_n)$ span $T(\mathbf{R} \times \mathbf{R}^n)$, this implies the desired result.

The proof is basically by induction on m . Recall from Theorem 3.8 that $M^\beta D^3 u \in C(\mathbf{R}; H_{\text{loc}}^{s-3}(\mathbf{R}^n)) \cap C^1(\mathbf{R}; H_{\text{loc}}^{s-4}(\mathbf{R}^n))$, $M^\beta (D^2 \varphi) \in C(\mathbf{R}; H_{\text{loc}}^{s-4}(\mathbf{R}^{n-1}))$ for all β . For $m = 0$ and $\alpha = 0$, since $s > \frac{n}{2} + 4$, $\varphi(t, x') \in C^2(\mathbf{R} \times \mathbf{R}^{n-1})$ and so $v, w \in C(\mathbf{R} \times \mathbf{R}^n)$. Suppose that $M^\alpha v, M^\alpha w \in C(\mathbf{R} \times \mathbf{R}^n)$ for $|\alpha| = k - 1$. Then

$$(4.3) \quad M_j M^\alpha v = D_0 M_j M^\alpha D^3 u + [M_j M^\alpha, D_0] D^3 u \in C(\mathbf{R} \times \mathbf{R}^n).$$

By Lemma 3.4, $[M_0^k, D_0] = \sum_{j=0}^{k-1} c_j(k) D_0 M_0^j$, where $c_j(k)$ are constants depending only on k . Therefore, $M^\alpha v \in C(\mathbf{R} \times \mathbf{R}^n)$ for $|\alpha| = k$. Similarly, $M^\alpha w \in C(\mathbf{R} \times \mathbf{R}^n)$.

In order to see the induction process more precisely, we prove (4.2) for $m = 1$. From (3.15) and the equation $P_2(D^3 u) = G(DD^3 u)$, v and w satisfy

equations of the form:

$$(4.4) \quad \begin{aligned} D_0 w + r_{00} M_0 v + r_{11} M_1 w + \sum_{j=2}^n r_{0j} M_j v + \sum_{j=2}^n r_{1j} M_j w + \sum_{i,j=2}^n r_{ij} M_i M_j D^3 u \\ = f(D^2 u, v, w, MD^3 u, D^2 \varphi) \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} D_1 v + r_{00} M_0 v + r_{11} M_1 w + \sum_{j=2}^n r_{0j} M_j v + \sum_{j=2}^n r_{1j} M_j w + \sum_{i,j=2}^n r_{ij} M_i M_j D^3 u \\ = f(D^2 u, v, w, MD^3 u, D^2 \varphi), \end{aligned}$$

where f is a smooth function of its arguments. Therefore, by the case of $m = 0$, $D_0 w$ and $D_1 v$ are in the space $C(\mathbf{R} \times \mathbf{R}^n)$.

We apply M_i to equations (4.4) and (4.5). Then we have

$$\begin{aligned} M_i D_0 w &= M_i D_1 v \\ &= (M_i r_{00}) M_0 v + r_{00} M_i M_0 v + (M_i r_{11}) M_1 w + r_{11} M_i M_1 w \\ &\quad + \sum_j (M_i r_{0j}) M_j v + r_{0j} M_i M_j v + \sum_j (M_i r_{1j}) M_j w + r_{1j} M_i M_j w \\ &\quad + \sum_{j,k} (M_i r_{jk}) M_j M_k D^3 u + r_{jk} M_i M_j M_k D^3 u \\ &\quad + g(M_i D^2 u, M_i v, M_i w, M_i MD^3 u, M_i D^2 \varphi), \end{aligned}$$

where g is a smooth function of its arguments. Therefore, by (4.3) and since $M_i D^2 \varphi \in C(\mathbf{R}; H^{s-4}(\mathbf{R}^n))$, we have $M_i D_1 v, M_i D_0 w \in C(\mathbf{R} \times \mathbf{R}^n)$ for all $i = 0, 1, \dots, n$. By an induction argument and improved piecewise regularity of φ with respect to M_i , we have

$$M^\alpha D_1 v, M^\alpha D_0 w \in C(\mathbf{R} \times \mathbf{R}^n) \quad \text{for all } \alpha.$$

In order to show that $D_0 v$ is piecewise continuous, we apply D_0 to equation (4.5) to obtain the equation

$$\begin{aligned} D_1(D_0 v) + (D_0 r_{00}) M_0 v + r_{00} M_0 D_0 v + (D_0 r_{11}) M_1 w + r_{11} M_1 D_0 w \\ + \sum_j (D_0 r_{0j}) M_j v + r_{0j} M_j D_0 v + \sum_j (D_0 r_{1j}) M_j w + r_{1j} M_j D_0 w \\ + \sum_{j,k} (D_0 r_{jk}) M_j M_k D^3 u + r_{jk} M_j M_k D_0 D^3 u \\ = f_1(D_0 D^2 u, D_0 v, D_0 w, MD_0 D^3 u, D_0 D^2 \varphi), \end{aligned}$$

or

$$(4.6) \quad Q_+(D_0 v) + g_0(D^2 u, v, w, MD^3 u, D^2 \varphi)(D_0 v) = g_1,$$

where $Q_+ = D_1 + r_{00} M_0 + \sum_{j=2}^n r_{0j} M_j$, g_0 is a smooth function of its arguments and g_1 is a smooth function of its arguments $D_0 D^2 u$, $D_0 D^2 \varphi$, $M_0 v$, $M_1 w$, $M_1 D_0 w$, $M_j v$, $M_j w$, $M_j D_0 w$, $M_i M_j D^3 u$ and $M_i M_j v$. By applying D_0 to

equation (3.21), we have the equation of form (3.22) replaced L and \tilde{L} by D_i and $\frac{1}{\psi_i}\partial_t$. Therefore, we have

$$(4.7) \quad Q(D_0 D^2 \varphi^1) = G_1(D_0 D^2 \varphi^1) + G_2,$$

where $Q = \partial_t - f'(Du(t, \varphi, x'), \tilde{D}\varphi) \sum_{j=2}^n \partial_{x_j}$ and G_1, G_2 are smooth functions of their arguments. Since Q is transversal to Σ_0, Σ_1 and $\{t=0\}$ by the slightly changed time coordinate, $D_0 D^2 \varphi$ is piecewise continuous by Lemma 4.2. Here we use the fact that piecewise continuous functions form an algebra invariant under composition with C^∞ functions. Therefore, (4.6) can be written as

$$(4.8) \quad Q_+(D_0 v) + (p.c.)(D_0 v) = p.c.$$

We note that Q_+ is a vector field transverse to Σ_1 and $t=0$. Since $t=0$ is a noncharacteristic hypersurface, all derivatives may be computed from the initial data at $t=0$ and thus $D_0 v(0, x)$ is p.c. Therefore, by Lemma 4.2, $D_0 v$ is p.c. on $\mathcal{O} \setminus \Sigma_0$. A similar proof for $Q_- = D_0 + r_{11}M_1 + \sum_{j=2}^n r_{1j}M_j$ shows that $D_1 w$ is p.c. on $\mathcal{O} \setminus \Sigma_1$.

In order to show that $D_0 M_i v$ and $D_1 M_i w$ are p.c. for $i = 0, 1, \dots, n$, we differentiate (4.6) with respect to M_i . Then we have

$$(4.9) \quad Q_+(M_i D_0 v) + (h_0)(M_i D_0 v) = (h_1),$$

where h_0 and h_1 are smooth functions of their arguments, which are all p.c. Note that $D_0 M_i D^2 \varphi$ is piecewise continuous by similar arguments as above after applying M_i to equation (4.7). $D_0 M_i v$ satisfies an initial value problem of the form (4.11), and then, by Lemma 4.2, $D_0 M_i v$ is p.c. on $\mathcal{O} \setminus \Sigma_0$. A similar proof for $Q_- = D_0 + r_{11}M_1 + \sum_{j=2}^n r_{1j}M_j$ shows that $D_1 M_i w$ is p.c. on $\mathcal{O} \setminus \Sigma_1$. Again, by an induction argument and improved piecewise regularity of φ with respect to D_0, D_1 and M_i , we have

$$M^\alpha D_0 v, M^\alpha D_1 w \in C(\mathbf{R} \times \mathbf{R}^n) \quad \text{for all } \alpha.$$

We assume that (4.2) and $D_0^i D_1^j M^\alpha v, D_0^i D_1^j M^\alpha w$, hold for $i+j = m-1$. Then we will show that (4.2) holds for m . Using the induction hypothesis, it follows from equations (4.4) and (4.5) that

$$D_0^i D_1^j M^\alpha w \in p.c. \quad \text{if } i \geq 1,$$

$$D_0^i D_1^j M^\alpha v \in p.c. \quad \text{if } j \geq 1.$$

Therefore, it remains to show that $D_0^m M^\alpha v$ and $D_1^m M^\alpha w$ are p.c. We will work on the v terms: the w terms are treated similarly.

By applying D_0^{m-1} to (4.6) and using the induction hypothesis, we obtain an equation

$$(4.10) \quad Q_+(D_0^m v) + (p.c.)(D_0^m v) = p.c.$$

Therefore, by a similar argument to that given in the case $m=1$, $D_0^m v$ is p.c. on $\mathcal{O} \setminus \Sigma_0$. A similar proof for Q_- shows that $D_1^m w$ is p.c. on $\mathcal{O} \setminus \Sigma_1$. This completes the proof of (4.2) for $\alpha=0$.

In order to consider the case $|\alpha|=1$, we differentiate (4.10) with respect to M_i noting that the p.c. terms remain p.c. Note that by the arguments as above

piecewise regularity of φ with respect to D_0 and M_i is inductively improved. Then we have

$$Q_+(M_i D_0^m v) + (H_0)(M_i D_0^m v) = H_1.$$

$M_i D_0^m v$ satisfies an initial value problem of the form (4.11). Therefore, by Lemma 4.2, we can conclude that $D_0^m M_i v$ is p.c. on $\mathcal{O} \setminus \Sigma_0$. Similarly, $D_1^m M_i w$ is p.c. on $\mathcal{O} \setminus \Sigma_1$ and we have proven (4.2) in the case $|\alpha| = 1$. The general case is proved in the same method by induction on $|\alpha|$. The proof of the theorem is completed, except for Lemma 4.2.

Lemma 4.2. *Suppose that $A(t, x)$ and $F(t, x)$ are p.c. on $\mathcal{O} \setminus \Sigma_0 \cup \Sigma_1$. Then the initial value problem for the linear equation*

$$(4.11) \quad Q_+ z(t, x) + A(t, x)z(t, x) = F(t, x)$$

has a unique piecewise continuous solution on $\mathcal{O} \setminus \Sigma_0$ if $z(0, x)$ is piecewise continuous on $\mathcal{O}_0 \setminus \Gamma$. Here Q_+ is a first order operator with piecewise continuous coefficients as described after (4.6).

Proof. The argument is similar to that in Rauch and Reed [15] except that the coefficients of Q_+ are merely assumed to be piecewise continuous instead of smooth.

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